Weyl Group Multiple Dirichlet Series I

Benjamin Brubaker, Daniel Bump, Gautam Chinta,
Solomon Friedberg and Jeffrey Hoffstein

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Given a root system Φ of rank r and a global field F containing the n-th roots of unity, it is possible to define a *Weyl group multiple Dirichlet series* whose coefficients are n-th order Gauss sums. It is a function of r complex variables, and it has meromorphic continuation to all of $\mathbb{C}^r$, with functional equations forming a group isomorphic to the Weyl group of Φ. *Weyl group multiple Dirichlet series and their residues unify many examples that have been studied previously in a case-by-case basis, often with applications to analytic number theory.* (Examples may be found in the final section of the paper.)

We believe these Weyl group multiple Dirichlet series are fundamental objects. The goal of this paper is to define these series for any such Φ and F, and to indicate how to study them. We will note the following points.

- The coefficients of the Weyl group multiple Dirichlet series are multiplicative, but the multiplicativity is *twisted*, so the Dirichlet series is not an Euler product.

- Due to the multiplicativity, description of the coefficients reduces to the case where the parameters are powers of a single prime p. There are only finitely many such coefficients (for given p).

- In the "stable case" where n is sufficiently large (depending on Φ), the number of nonzero coefficients in the p-part is equal to the order of the Weyl group. Indeed, these nonzero coefficients are parametrized in a natural way by the Weyl group elements.

- The p-part coefficient parametrized by a Weyl group element w is a product of $l(w)$ Gauss sums, where l is the length function on the Weyl group.

We note a curious similarity between this description and the coefficients of the generalized theta series on the n-fold cover of $\mathrm{GL}(n)$ and $\mathrm{GL}(n-1)$; these coefficients are determined in Kazhdan and Patterson [17] and discussed further in Patterson [21]. See Bump and Hoffstein [11] or Hoffstein [15] for a "classical" description of these coefficients. The noted similarity means that the complete Mellin transform of the theta function would be a multiple Dirichlet series resembling our A_{n+1} multiple Dirichlet series. There is no *a priori* reason that we are aware of for the complete Mellin transform of the generalized theta series to have meromorphic continuation. For example if $n = 4$, the complete Mellin transform of an $\mathrm{GL}(4)$ cusp form (nonmetaplectic) would have a meromorphic continuation only in the special cases $s_1 = s_2 + s_3$ or $s_3 = s_1 + s_2$, in which case it produces a product of L-functions by Bump and Friedberg [9]. (If s_1, s_2 and s_3 are in general position, meromorphic continuation fails due to the Estermann phenomenon.) This observation raises quite a few potentially interesting questions.

The Weyl group multiple Dirichlet series are expected to be Whittaker coefficients of metaplectic Eisenstein series, though we will not prove that here; we will, however, come back to it in a later paper. The claim that the Whittaker coefficients of metaplectic Eisenstein series have such a simple structure appears to be new, and is essentially global in nature, because the representations of the metaplectic covers of semisimple groups do not in general have unique Whittaker models. In this paper we will study the Weyl group multiple Dirichlet series without making use of Eisenstein series on higher-rank metaplectic groups. However Eisenstein series on the n-fold cover of SL_2 underlie the functional equations of the Kubota Dirichlet series that are the basic building blocks in our construction. Our methods are those laid out in Bump, Friedberg and Hoffstein [10], applying a theorem of Bochner [1] from several complex variables to reduce everything directly to the case of Kubota's Dirichlet series. (The use of Bochner's theorem would also be implicit in an approach based on higher-rank Eisenstein series, in the reduction of the functional equations to rank one; this type of argument goes back to Selberg [22]. But an approach based on the theory of higher rank metaplectic Eisenstein series would be considerably more difficult.)

We will begin our treatment with a heuristic formulation, in which important aspects of the true situation are ignored to obtain some intuition. We will thus gain heuristics that predict the (twisted) multiplicativity of the coefficients and the group of automorphisms of $\mathbb{C}^r$ comprising the group of functional equations; it is isomorphic to the Weyl group of Φ. Although we will not discuss it much in this paper, the heuristic viewpoint is also useful for inferring facts about residues of Weyl group multiple Dirichlet series.

Although the heuristic point of view is unrigorous, we will proceed to a completely rigorous formulation. Our discussion will thus have three stages. The first stage is the heuristic formulation. In the second stage, we will completely describe the p-part of the "stable" Weyl group multiple Dirichlet series; this is accomplished in Section 2. The third stage, completing the theory, requires careful bookkeeping with Hilbert symbols. Stages 1 and 2 are carried out completely here; for the third stage, we carry it out here for the A_2 Weyl group multiple Dirichlet series, and for general Φ in [5].

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1 "Heuristic" Multiple Dirichlet Series

The paper of Brubaker and Bump [4] will be our general reference for most foundational matters; particularly, the properties of Gauss sums, Hilbert symbols and power residue symbols that we need are there. For root systems, see Bourbaki [2] or Bump [8].

A *root system* is a finite subset Φ of Euclidean space $\mathbb{R}^r$ of nonzero vectors such that if $\alpha \in \Phi$, and if $\sigma_\alpha : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ is the reflection in the hyperplane through the origin perpendicular to the vector α then $\sigma_\alpha(\Phi) = \Phi$, and if $\alpha, \beta \in \Phi$, then $\beta - \sigma_\alpha(\beta)$ is an integer multiple of α. Since $-\alpha = \sigma_\alpha(\alpha)$, these axioms imply that $-\alpha \in \Phi$. The root system is called *reduced* if α and 2α are not both in Φ, and it is called *irreducible* if it is not the union of two smaller root systems that span orthogonal subspaces of $\mathbb{R}^r$. The root system Φ is called *simply-laced* if all roots have the same length.

We choose a partition of Φ into subsets Φ^+ and Φ^- of *positive* and *negative* roots such that for some hyperplane H through the origin, the roots in Φ^+ all lie on one side of H, and the roots in Φ^- lie on the other side. A positive root $\alpha \in \Phi^+$ is called *simple* if it cannot be written as a sum of other positive roots.

Let Φ be a reduced root system in $\mathbb{R}^r$, and let Δ denote the set of simple positive roots. The *Weyl group* W of Φ is the group generated by the σ_α such that $\alpha \in \Phi$. It is also generated by the σ_α with $\alpha \in \Delta$. Let

$$\Delta = \{\alpha_1, \cdots, \alpha_r\}$$

be the set of simple positive roots, and denote $\sigma_i = \sigma_{\alpha_i}$. Then W has a presentation consisting of the relations

$$\sigma_i^2 = 1, \qquad (\sigma_i \sigma_j)^{2r(\alpha_i,\alpha_j)} = 1,$$

where, if θ is the angle between the roots,

$$r(\alpha,\beta)=\begin{cases}0 & \text{if } \alpha,\beta \text{ are orthogonal},\\ 1 & \text{if } \theta=\frac{2\pi}{3},\\ 2 & \text{if } \theta=\frac{3\pi}{4},\\ 3 & \text{if } \theta=\frac{5\pi}{6}.\end{cases} \tag{1}$$

Thus W is a Coxeter group.

Fix $n>1$, and let F be an algebraic number field containing the group μ_n of n-th roots of unity in $\mathbb{C}$. We will also assume that -1 is an n-th power. It follows that F is totally complex. Let S be a finite set of places including all the infinite ones, those dividing n, all those that are ramified over $\mathbb{Q}$ and enough others that the ring $\mathfrak{o}_S$ of S-integers is a principal ideal domain. We recall that $\mathfrak{o}_S$ is the set of elements $\alpha\in F$ such that $|\alpha_v|_v\leqslant 1$ for all places v of F not in S. We will denote

$$F_\infty=\prod_{v\in S_\infty}F_v,\qquad F_{\text{fin}}=\prod_{v\in S_{\text{fin}}}F_v,\qquad F_S=\prod_{v\in S}F_v=F_\infty\times F_{\text{fin}},$$

where S_∞ is the set of archimedean places in S, and S_{fin} is the set of nonarchimedean ones. We embed $\mathfrak{o}_S$ in F_S along the diagonal. It is discrete and cocompact.

If v is a place of F, the Hilbert symbol is a map $F_v^\times\times F_v^\times\longrightarrow\mu_n$, denoted

$$c,d\longmapsto(c,d)_v,\qquad c,d\in F_v^\times.$$

We will also denote

$$(c,d)=\prod_{v\in S}(c_v,d_v)_v.$$

The symbol $(\ ,\ )$ is a skew-symmetric bilinear pairing on $F_S^\times$ whose properties are discussed in Brubaker and Bump [4].

If c and d are nonzero elements of $\mathfrak{o}_S$, let $\left(\frac{c}{d}\right)$ denote the power residue symbol. Its properties are discussed in Brubaker and Bump [4]. We mention that it is multiplicative in both c and d, depends only on c modulo d, and also depends only on the ideal generated by d. Most important is the *reciprocity law*

$$\left(\frac{c}{d}\right)=(d,c)\left(\frac{d}{c}\right). \tag{2}$$

If $0\neq c\in\mathfrak{o}_S$, and $t\in\mathbb{Z}$ let

$$g_t(\alpha,c)=\sum_{d \bmod c}\left(\frac{d}{c}\right)^t\psi\left(\frac{\alpha d}{c}\right),$$

where $\left(\frac{c}{d}\right)$ is the power residue symbol and ψ is a nontrivial additive character of F_S whose conductor is precisely $\mathfrak{o}_S$; since $\mathfrak{o}_S$ is a principal ideal domain, such a character always exists. The properties of Gauss sums are summarized in Brubaker and Bump [4]. If $t = 1$ we may simply denote $g_1(\alpha, c) = g(\alpha, c)$.

Let Φ be a reduced root system. For each α_i we choose a complex variable $s_{\alpha_i} = s_i$. We will define a multiple Dirichlet series $Z_\Psi(s) = Z_\Psi(s_1, \cdots, s_r)$, which will be a function of r complex variables. It will depend on an extra datum Ψ that we will eventually describe, but first we give a rough "heuristic" description of Z_Ψ. The heuristic description will be incorrect but suitable for fixing ideas. In discussing the heuristic form the datum Ψ is not too important, and we suppress it from the notation. It will be restored when we move past the heuristic form to a correct definition of Z_Ψ.

In this Section we will make an assumption that is unrealistic but convenient for heuristic purposes. It will be seen in our discussions that both Hilbert symbols and power residues symbols appear; the power residue symbols are essential, but the Hilbert symbols are only needed for bookkeeping purposes. A lot can be inferred by ignoring them. We will therefore pretend that the symbol (c, d) is trivial, and that reciprocity is perfect:

$$\left(\frac{c}{d}\right) = \left(\frac{d}{c}\right).$$

Then the heuristic form of the Weyl group multiple Dirichlet series is

$$Z(s) = \sum_{\substack{c_\alpha \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (\alpha \in \Delta)}} g_\alpha(c_\alpha) \left[\prod_{\alpha,\beta} \left(\frac{c_\alpha}{c_\beta}\right)^{-r(\alpha,\beta)} \right] \prod_{\alpha \in \Delta} \mathbb{N}(c_\alpha)^{-2s_\alpha}.$$

where the product is over pairs of simple roots α and β, and notation is as follows. Due to our assumption on reciprocity, it does not matter whether we take the pair α, β or β, α, but we consider these to be the same pair, so there are $\frac{1}{2}r(r-1)$ factors in the product. The Gauss sum is

$$g_\alpha(m) = \begin{cases} g_1(1, m) & \text{if } \alpha \text{ is a short root,} \\ g_2(1, m) & \text{if } \alpha \text{ is a long root and } \Phi \neq G_2. \\ g_3(1, m) & \text{if } \alpha \text{ is a long root and } \Phi = G_2, \end{cases}$$

and $r(\alpha, \beta)$ is defined by (1). The absolute norm $\mathbb{N}(c_\alpha)$ is the cardinality of $\mathfrak{o}_S/c_\alpha\mathfrak{o}_S$.

There is also a normalizing factor, $N(s) = N(s_1, \cdots, s_r)$. We will describe it more precisely later; for the time being, let us only state that it is a product of zeta functions and Gamma functions. We denote the normalized Dirichlet series as

$$Z^*(s) = N(s)\, Z(s).$$

There are a couple of things that are wrong with this description. First, we have made the unrealistic assumption of perfect reciprocity; we have written the sum as if each term depends only on the ideal of c_α, whereas what we have written will change by a Hilbert symbol if c_α is multiplied by a unit; and, most seriously, we have only described the coefficient in the Dirichlet series in the very special case where the c_α are coprime.

Despite these defects, the heuristic Dirichlet series is useful for deducing properties of the corrected version Z_Ψ, which we will come to later. So we will draw what conclusions we can from the heuristic form. The defects can all be fixed, as we will eventually see.

We will make use of the functional equations of *Kubota Dirichlet series*, which are the Dirichlet series formed with Gauss sums. Let

$$\mathcal{D}(s,\alpha) = \sum_{0 \neq c \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g(\alpha, c)\ \mathbb{N}(c)^{-2s}\ .$$

In writing this there is again an unrealistic assumption, since the summand is actually not invariant under the action of units – but at the moment, we recall, we are pretending that the Hilbert symbol is trivial, and accepting this fantasy $g(\alpha, \varepsilon c) = g(\alpha, c)$ when $\varepsilon \in \mathfrak{o}_S^\times$. Let

$$\mathcal{D}^*(s,\alpha) = \boldsymbol{G}_n(s)^{\frac{1}{2}[F:\mathbb{Q}]}\ \zeta_F(2ns-n+1)\, \mathcal{D}(s,\alpha),$$

where

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

The exponent $\frac{1}{2}[F:\mathbb{Q}]$ is just the number of archimedean places of the totally complex field F. By the multiplication formula for the Gamma function,

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-\frac{1}{2})}\ n^{-1/2+n(2s-1)} \prod_{j=1}^{n-1} \Gamma\left(2s-1+\frac{j}{n}\right).$$

Then $\mathcal{D}^*$ has a functional equation, due to Kubota, which says (essentially)

$$\mathcal{D}^*(s,\alpha) = \mathbb{N}(\alpha)^{1-2s}\, \mathcal{D}^*(1-s,\alpha). \tag{3}$$

Once again, there are some problems to correct – the Dirichlet series $\mathcal{D}$ has not been defined correctly, and the functional equation actually involves a finite scattering matrix. See Section 3 (and [4]) for the correct definition and functional equation.

More generally, let $0 < t \in \mathbb{Z}$ be given, and let

$$\mathcal{D}_t(s, \alpha) = \sum_{0 \neq c \in \mathfrak{o}_S/\mathfrak{o}_S^\times} g_t(\alpha, c)\ \mathbb{N}(c)^{-2s}\ .$$

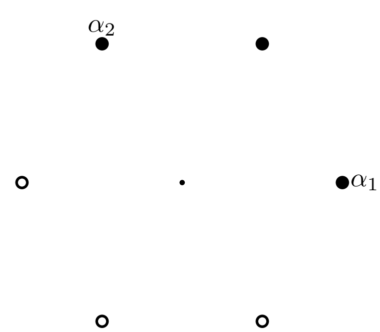
We define

$$\mathcal{D}_t^*(s, \alpha) = \boldsymbol{G}_m(s)^{\frac{1}{2}[F:\mathbb{Q}]}\ \zeta_F(2ms - m + 1)\, \mathcal{D}_t(s, \alpha), \qquad m = \frac{n}{\gcd(n, t)}. \tag{4}$$

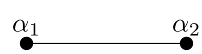
This value of m appears since g_t is an m-th order Gauss sum.

The heuristic definition is sufficient to predict the variable changes for the functional equations that Z will satisfy. It also predicts the normalizing factor of Z. We illustrate these points with two examples, one simply-laced, the other not.

As a first example, consider the root system of Cartan type A_2, whose Weyl group is isomorphic to the symmetric group S_3. There are two simple positive roots α_1 and α_2, and the root system looks like this:



The three positive roots are marked in black, the three negative ones in white. The roots α_1 and α_2 make an angle of $\frac{2\pi}{3}$. These facts can also be read off from the Dynkin diagram:



We see that

$$Z(s_1, s_2) = \sum_{c_1, c_2} g(1, c_1)\, g(1, c_2) \left(\frac{c_1}{c_2}\right)^{-1} \mathbb{N}(c_1)^{-2s_1}\, \mathbb{N}(c_2)^{-2s_2}. \tag{5}$$

Since $|g(1, c_1)| = \mathbb{N}c_1^{1/2}$, this series is absolutely convergent in the region

$$\Lambda_0 = \left\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{re}(s_1), \mathrm{re}(s_2) > \frac{3}{4}\right\}.$$

Let us first consider the functional equation with respect to s_1. As we will see, this functional equation has the form

$$\sigma_1 = \begin{cases} s_1 \longmapsto 1 - s_1, \\ s_2 \longmapsto s_1 + s_2 - \frac{1}{2}. \end{cases} \tag{6}$$

We have

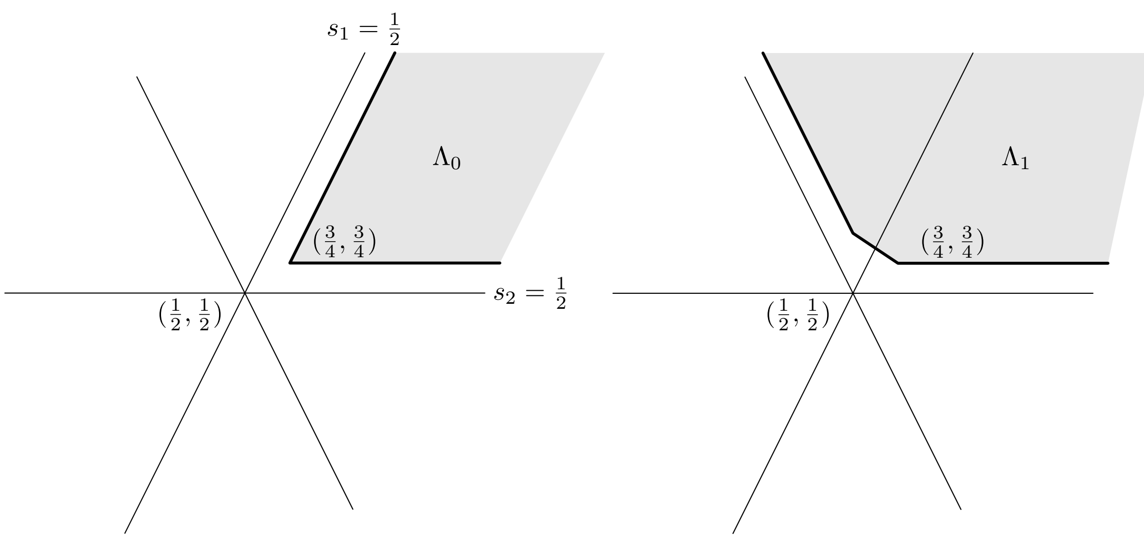
$$g(1, c_1)\left(\frac{c_1}{c_2}\right)^{-1} = g(c_2, c_1),$$

so

$$\begin{aligned} Z(s_1, s_2) &= \\ \sum_{c_2} g(1, c_2)\left[\sum_{c_1} g(c_2, c_1)\mathbb{N}(c_1)^{-2s_1}\right]\mathbb{N}(c_2)^{-2s_2} &= \\ \sum_{c_2} g(1, c_2)\,\mathcal{D}(s_1, c_2)\,\mathbb{N}(c_2)^{-2s_2}. \end{aligned}$$

Now it is expected that this expression has analytic continuation a larger region than the original sum; indeed, this representation gives continuation to this region

$$\Lambda_1 = \left\{(s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{re}(s_2), \mathrm{re}\left(s_1 + s_2 - \frac{1}{2}\right) > \frac{3}{4}, \mathrm{re}\left(s_1 + 2s_2 - \frac{1}{2}\right) > \frac{7}{4}\right\}.$$



In this figure, illustrating Λ_0 and Λ_1, we are representing the complex pair (s_1, s_2) by its real part $(\mathrm{re}(s_1), \mathrm{re}(s_2))$. We have tilted the s_2 axis so that σ_1 is a rigid motion; it is the reflection in the marked line $s_1 = \frac{1}{2}$. The region Λ_1 is the convex hull of $\Lambda_0 \cup \sigma_1 \Lambda_0$.

There is, similarly, a functional equation

$$\sigma_2 = \begin{cases} s_1 \longmapsto s_1 + s_2 - \frac{1}{2}, \\ s_2 \longmapsto 1 - s_2, \end{cases} \tag{7}$$

which is the reflection in the other marked line $s_2 = \frac{1}{2}$. This gives analytic continuation to the region

$$\Lambda_2 = \left\{ (s_1, s_2) \in \mathbb{C}^2 \mid \mathrm{re}(s_1), \mathrm{re}\left(s_1 + s_2 - \frac{1}{2}\right) > \frac{3}{4}, \mathrm{re}\left(2s_1 + s_2 - \frac{1}{2}\right) > \frac{7}{4} \right\}. \tag{8}$$

Together the transformations σ_1 and σ_2 generate the A_2 Weyl group, isomorphic to the symmetric group S_3. The analytic continuation to all (s_1, s_2) now follows by an argument based on Bochner's Theorem. See Theorem 2 below.

We can now prescribe the normalizing factor $N(s_1, s_2)$. It is

$$\boldsymbol{G}_n(s_1)\, \boldsymbol{G}_n(s_2)\, \boldsymbol{G}_n\left(s_1 + s_2 - \frac{1}{2}\right) \times \zeta_F(2ns_1 - n + 1)\, \zeta_F(2ns_2 - n + 1)\, \zeta_F(2ns_1 + 2ns_2 - 2n + 1).$$

With this factor we have

$$Z^*(s_1, s_2) = \boldsymbol{G}_n(s_2)\, \boldsymbol{G}_n\left(s_1 + s_2 - \frac{1}{2}\right) \zeta_F(2ns_2 - n + 1)\, \zeta_F(2ns_1 + 2ns_2 - 2n + 1) \times \sum_{c_2} g(1, c_2)\, \mathcal{D}^*(s_1, c_2)\, \mathbb{N}(c_2)^{-2s_2}.$$

Now the functional equation (6) is perfect – the two factors

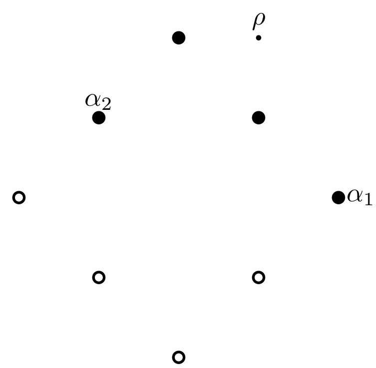
$$\boldsymbol{G}_n(s_2) \zeta_F(2ns_2 - n + 1) \qquad \text{and} \qquad \boldsymbol{G}_n\left(s_1 + s_2 - \frac{1}{2}\right) \zeta_F(2ns_1 + 2ns_2 - 2n + 1)$$

are interchanged, and the third factor has been absorbed into $\mathcal{D}^*$. Note that in the functional equation (3) the series $\mathcal{D}^*(s_1, c_2)$ is related to $\mathbb{N}(c_2)^{1-2s_1} \mathcal{D}^*(1 - s_1, c_2)$, so the Dirichlet series is transformed into

$$\sum_{c_2} g(1, c_2)\, \mathcal{D}^*(1 - s_1, c_2)\, \mathbb{N}(c_2)^{1-2s_1-2s_2},$$

from which we get (6).

Next let us consider an example that is not simply laced. We consider the B_2 root system, which looks like this:



In this example, we have the heuristic series

$$Z(s_1, s_2) = \sum_{c_1, c_2} g_2(c_1)\, g_1(c_2) \left(\frac{c_1}{c_2}\right)^{-2} \mathbb{N}(c_1)^{-2s_1}\, \mathbb{N}(c_2)^{-2s_2}.$$

We can write

$$Z(s_1, s_2) = \sum_{c_2} g_1(c_2) \left[\sum_{c_1} g_2(c_2, c_1) \mathbb{N}(c_1)^{-2s_1}\right] \mathbb{N}(c_2)^{-2s_2} = \sum_{c_2} g_1(c_2)\, \mathcal{D}_2(s_1, c_2)\, \mathbb{N}(c_2)^{-2s_2},$$

which gives us the functional equation

$$\begin{cases} s_1 \longmapsto 1 - s_1, \\ s_2 \longmapsto s_1 + s_2 - \frac{1}{2}. \end{cases}$$

More interestingly, if we write

$$Z(s_1, s_2) = \sum_{c_1} g_2(c_1) \left[\sum_{c_2} g_1(c_1^2, c_2)\, \mathbb{N}(c_2)^{-2s_2}\right] \mathbb{N}(c_1)^{-2s_1} = \sum_{c_1} g_2(c_1)\, \mathcal{D}(s_2, c_1^2)\, \mathbb{N}(c_1)^{-2s_1},$$

from which we deduce the functional equation

$$\begin{cases} s_1 \longmapsto s_1 + 2s_1 - 1, \\ s_2 \longmapsto 1 - s_2. \end{cases}$$

The two functional equations generate a group isomorphic to the Weyl group of Φ, which is of order 8.

In this example, there is a difference between the case where n is even and the case where n is odd. Although the group of functional equations is independent of the parity of n, the normalizing factor is *dependent*. This is because of (4). When $t = 2$, the factor $m = n/\gcd(2, n)$ is needed for the factor $\mathcal{D}_2$ coming from the long root. The normalizing factor is

$$\boldsymbol{G}_n(s_1)\zeta_F(2ns_1 - n + 1)\,\boldsymbol{G}_n(s_1 + s_2 - \frac{1}{2})\zeta_F(2ns_1 - 2n + 1)$$
$$\boldsymbol{G}_m(s_2)\zeta_F(2ms_2 - m + 1)\,\boldsymbol{G}_m(2s_1 + s_2 - 1)\zeta_F(4ms_1 + 2ms_2 - 2m + 1),$$

and the meaning of m is dependent on the parity of n.

Although the "heuristic" Dirichlet series is too unrealistic to be a perfect guide, we have just seen that it can predict the group of functional equations. It can also predict the *multiplicativity* of the coefficients, as we will now consider.

Returning to the A_2 example to explain this point, the heuristic form (5) is a stand-in for an actual Dirichlet series

$$Z_\Psi(s_1, s_2) = \sum_{c_1, c_2} H(c_1, c_2)\,\Psi(c_1, c_2)\,\mathbb{N}(c_1)^{-2s_1}\,\mathbb{N}(c_2)^{-2s_2}. \tag{9}$$

The factor Ψ can be ignored for the time being; in this section we write

$$Z(s_1, s_2) = \sum_{c_1, c_2} H(c_1, c_2)\ \mathbb{N}(c_1)^{-2s_1}\,\mathbb{N}(c_2)^{-2s_2}.$$

The coefficients $H(c_1, c_2)$ will have a "twisted" multiplicativity. True multiplicativity would be the statement that if $\gcd(c_1c_2, c_1'c_2') = 1$ then

$$H(c_1c_1', c_2c_2') = H(c_1, c_2)\,H(c_1', c_2').$$

This is not true. Instead, we have

$$H(c_1c_1', c_2c_2') = H(c_1, c_2)H(c_1', c_2')\left(\frac{c_1}{c_1'}\right)\left(\frac{c_1'}{c_1}\right)\left(\frac{c_2}{c_2'}\right)\left(\frac{c_2'}{c_2}\right)\left(\frac{c_1}{c_2'}\right)^{-1}\left(\frac{c_1'}{c_2}\right)^{-1}. \tag{10}$$

Of course we are currently pretending that all Hilbert symbols are trivial so that $\left(\frac{c}{c'}\right) = \left(\frac{c'}{c}\right)$; one might therefore write the right-hand side as

$$H(c_1, c_2)H(c_1', c_2') \left(\frac{c_1}{c_1'}\right)^2 \left(\frac{c_2}{c_2'}\right)^2 \left(\frac{c_1}{c_2'}\right)^{-1} \left(\frac{c_1'}{c_2}\right)^{-1}.$$

We have written (10) without this "simplification" since as written it is correctly stated, even without the simplifying assumption that all Hilbert symbols are trivial.

The multiplicativity (10) can be checked when all four parameters c_1, c_2, c_1' and c_2' are mutually coprime using the fact that

$$g(m, cc') = \left(\frac{c}{c'}\right) \left(\frac{c'}{c}\right) g(m, c)\, g(m, c'), \qquad \text{if } c, c' \text{ are coprime.} \tag{11}$$

In this case we have specified

$$H(c_1, c_2) = g(1, c_1)\, g(1, c_2) \left(\frac{c_1}{c_2}\right)^{-1}.$$

The most serious defect in the heuristic form of the multiple Dirichlet series is that we have only specified $H(c_1, c_2)$ when c_1, c_2, c_1' and c_2' are pairwise coprime. However we have made some progress towards giving the general recipe, since we have deduced the multiplicativity (10). It is a small leap to guess that this formula is correct assuming only that $\gcd(c_1c_2, c_1'c_2') = 1$. Given (10), we are reduced to specifying $H(c_1, c_2)$ when c_1 and c_2 are powers of the same prime p. As we will see in the following section, this question turns out to have a simple and beautiful answer if n is sufficiently large.

2 The Stable Case

As we have explained in Section 1, the "heuristic" formula for the Dirichlet series is sufficient to deduce the multiplicativity of the terms, which reduces their specification to that of the p-part, where p is a prime of $\mathfrak{o}_S$. By this, we mean the coefficients

$$H(p^{k_1}, \cdots, p^{k_r}). \tag{12}$$

We will specify these in this section.

As in Section 1, no Hilbert symbols will appear in this section. How the defects of the "heuristic" Dirichlet series are to be corrected has yet to be revealed, and will be taken up in the next section. But we are now outside the heuristic realm, and the formulas that we give for the p-part are exactly correct. The fact that no symbols appear in this section, yet the statements will require no further revision may appear surprising – see Remark 1 below for the explanation of this paradox.

There is an important caveat: we will give exact formulas for the terms (12), but these are only correct if n is sufficiently large. The meaning of "sufficiently large" is easiest to explain if Φ is simply laced. In this case, let α be the longest positive root, and write $\alpha = \sum_i d_i \alpha_i$ where, we recall, the α_i are the simple positive roots. In this case, if $n \geqslant \sum_i d_i$ then the formulas we give will be correct. We call this the *stable case*. (If Φ is not simply laced, see Brubaker, Bump and Friedberg [5] for the precise condition that n must satisfy for stability.)

In the unstable case (where n is small) the multiple Dirichlet series should exist and the nonzero coefficients that we describe will be present. However there will be other nonzero coefficients as well. We do not yet have a precise description of the terms (12) that is valid in the unstable case when Φ is an arbitrary root system. However a conjectural statement when $\Phi = A_r$ may be found in Brubaker, Bump, Friedberg and Hoffstein [6]. This conjectural description describes the coefficients as sums of products of Gauss sums indexed by Gelfand-Tsetlin patterns. It is proved correct when $r = 2$, or $n = 1$, and is consistent with results of Chinta [12] that describe the Weyl group multiple Dirichlet series for A_r when $r \leqslant 5$ and $n = 2$.

Define the support of H to be

$$\text{Supp}(H) = \{(k_1, \cdots, k_r) \,|\, H(p^{k_1}, \cdots, p^{k_r}) \neq 0\}.$$

It will be seen that this set, which does not depend on p, is finite, and in the stable case, is in bijection with the elements of the Weyl group.

Let

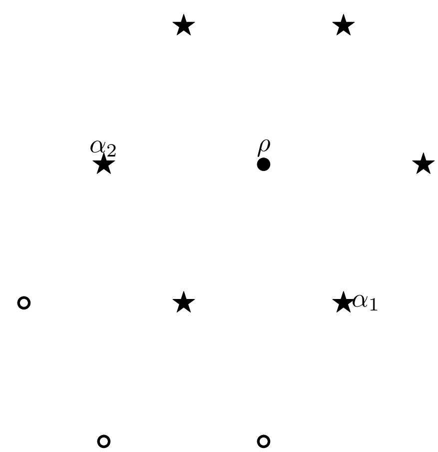
$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

If $w \in W$, the Weyl group, we have

$$\rho - w(\rho) = \sum_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \alpha.$$

These $|W|$ points form a figure that is congruent to supp(H). Before we give the general presecription, let us illustrate this point with a couple of examples.

First, if Φ is of type A_2, the points $\rho - w(\rho)$ are marked by stars in the following figure:



The black dots are the positive roots; two of them, at the simple roots α_1 and α_2, are obscured by stars. The white dots are the negative roots. It will be noted that the stars form a hexagon. Here, for comparison, are the nonzero values of $H(p^{k_1}, p^{k_2})$ for the A_2 Weyl group multiple Dirichlet series:

(k_1, k_2)	$H(p^{k_1}, p^{k_2})$
$(0,0)$	1
$(1,0)$	$g_1(1,p)$
$(0,1)$	$g_1(1,p)$
$(1,2)$	$g_1(1,p)g_1(p,p^2)$
$(2,1)$	$g_1(1,p)g_1(p,p^2)$
$(2,2)$	$g_1(1,p)^2\, g_1(p,p^2)$

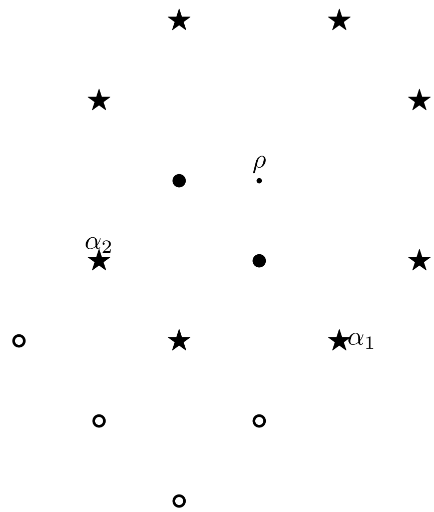
(13)

Thus

$$\mathrm{supp}(H) = \{(0,0),(1,0),(0,1),(1,2),(2,1),(2,2)\}$$

is a hexagon – exactly the shape of the figure of starred points $\rho - w(\rho)$. More precisely, the possible values of $k_1\alpha_1 + k_2\alpha_2$ are exactly the set of $\rho - w(\rho)$.

As a second example, which is not simply-laced, suppose that Φ is of type B_2. The points $\rho - w(\rho)$ are the starred vertices in the following diagram.



Again, the roots are labeled by dots (black for the positive roots, white for the negative ones) and the values of $\rho - w(\rho)$ are marked by stars. The nonzero values of $H(p^{k_1}, p^{k_2})$ are given by the following table:

k_2 \ k_1	0	1	2	3
0	1	$g_2(1,p)$		
1	$g_1(1,p)$		$g_1(1,p)g_2(p,p^2)$	
2				
3		$g_2(1,p)g_1(p^2,p^3)$		$g_2(1,p)g_2(p,p^2)$ $\times g_1(p^2,p^3)$
4			$g_1(1,p)g_2(p,p^2)$ $\times g_1(p^2,p^3)$	$g_2(1,p)g_1(1,p)$ $\times g_2(p,p^2)g_1(p^2,p^3)$

Thus

$$\mathrm{supp}(H) = \{(0,0),(1,0),(0,1),(2,1),(1,3),(3,3),(2,4),(3,4)\}$$

is precisely the set of (k_1, k_2) such that $k_1\alpha_1 + k_2\alpha_2$ can be expressed as $\rho - w(\rho)$ for some $w \in W$.

We will now describe the coefficients $H(p^{k_1}, \cdots, p^{k_r})$ in the stable case. If $\alpha \in \Phi$, we write

$$d(\alpha) = \sum_i c_i, \text{ where } \alpha = \sum_{\alpha_i \in \Delta} c_i \alpha_i.$$

Then we write

$$H(p^{k_1}, \cdots, p^{k_r}) = \prod_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^-}} g_\alpha(p^{d(\alpha)-1}, p^{d(\alpha)})$$

if there exists a $w \in W$ such that

$$\sum_{i=1}^{r} k_i \alpha = \rho - w(\rho),$$

while $H(p^{k_1}, \cdots, p^{k_r}) = 0$ if no such w exists.

Returning to the case $\Phi = B_2$, we embed the root system into $\mathbb{R}^2$ so that the simple roots and coroots are

$$\alpha_1 = (1, 0), \qquad \alpha_2 = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

The following table shows how the $H(k_1, k_2)$ are to be computed.

$w(\rho)$	$\rho - w(\rho)$	(k_1, k_2)	$\alpha \in \Phi^+, w(\alpha) \in \Phi^-$	$\prod g_\alpha(p^{d(\alpha)-1}, p^{d(\alpha)})$
$\left(\frac{1}{2}, 1\right)$	0	$(0,0)$	none	1
$\left(1, \frac{1}{2}\right)$	$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$(0,1)$	α_2	$g_1(1,p)$
$\left(-\frac{1}{2}, 1\right)$	$(1,0)$	$(1,0)$	α_1	$g_2(1,p)$
$\left(-1, \frac{1}{2}\right)$	$\left(\frac{3}{2}, \frac{1}{2}\right)$	$(2,1)$	$\alpha_2, \alpha_1+\alpha_2$	$g_1(1,p)\, g_2(p,p^2)$
$\left(\frac{1}{2}, -1\right)$	$(0,2)$	$(2,4)$	$\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2$	$g_1(1,p)\, g_2(p,p^2)$ $g_1(p^2,p^3)$
$\left(1, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$(1,3)$	$\alpha_1, 2\alpha_1+\alpha_2$	$g_2(1,p) g_1(p^2,p^3)$
$\left(-\frac{1}{2}, -1\right)$	$(1,2)$	$(3,4)$	$\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2$	$g_2(1,p)\, g_1(1,p)$ $g_2(p,p^2)\, g_1(p^2,p^3)$
$\left(-1, -\frac{1}{2}\right)$	$\left(\frac{3}{2}, \frac{3}{2}\right)$	$(3,3)$	$\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2$	$g_2(1,p)$ $g_2(p,p^2)\, g_1(p^2,p^3)$

3 The A_2 Weyl group multiple Dirichlet series

The Weyl group multiple Dirichlet series are, at this point of the paper, only partly defined. Coefficients $H(p^{k_1}, \cdots, p^{k_r})$ have been defined, but other aspects such as the multiplicativity have only been discussed under the unrealistic assumption that the Hilbert symbols can be ignored. We will give a completely rigorous discussion now of the case where $\Phi = A_2$, as an introduction to the more general case which will be treated in [5].

We begin by recalling the functional equations of Kubota Dirichlet series. The results of Kubota [18] were extended by Eckhardt and Patterson [13] and by Brubaker and Bump [4]. We consider [4] to be a companion piece and assume that the reader has it handy for reference.

Let F be an algebraic number field. As in the introduction we assume that F contains μ_n and that -1 is an n-th power in F, and other notations such as S, $\mathfrak{o}_S$, ψ, etc. will be as in the introduction.

We say a subgroup Υ of $F_S^\times$ is *isotropic* if the Hilbert symbol $(\varepsilon,\delta)=1$ for all $\varepsilon,\delta\in\Upsilon$. In particular, the group $\Omega=\mathfrak{o}_S^\times(F_S^\times)^n$ is maximal isotropic. Let $\mathcal{M}(\Omega)$ be the finite-dimensional vector space of functions Ψ on $F_{\text{fin}}^\times$ that satisfy

$$\Psi(\varepsilon c)=(\varepsilon,c)\,\Psi(c), \tag{14}$$

when $\varepsilon\in\Omega$. Note that if ε is sufficiently close to the identity in $F_S^\times$ it is an n-th power hence lies in Ω, so such a function is locally constant. The dimension of $\mathcal{M}(\Omega)$ is equal to the cardinality of $F_S^\times/\Omega$, which is finite. See Brubaker and Bump [4], Lemma 3.

If $\Psi\in\mathcal{M}(\Omega)$, define

$$\mathcal{D}(s,\Psi,\alpha)=\sum_{0\neq c\in\mathfrak{o}_S/\mathfrak{o}_S^\times} g(\alpha,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s}\,.$$

Here $\mathbb{N}(c)=|c|$ is the order of $\mathfrak{o}_S/c\mathfrak{o}_S$. The term $g(\alpha,c)\,\Psi(c)\,\mathbb{N}(c)^{-2s}$ is independent of the choice of representative c modulo $\mathfrak{o}_S^\times$ by (14) and the fact that if $\varepsilon\in\mathfrak{o}_S^\times$ we have

$$g(\alpha,\varepsilon c)=(c,\varepsilon)\,g(\alpha,c). \tag{15}$$

(See Brubaker and Bump [4] for details.) The normalized Kubota Dirichlet series is

$$\mathcal{D}^*(s,\Psi,\alpha)=\boldsymbol{G}_n(s)^{\frac{1}{2}[F:\mathbb{Q}]}\,\zeta_F(2ns-n+1)\,\mathcal{D}(s,\Psi,\alpha). \tag{16}$$

If $v\in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{o}_v$ is the local ring in F_v and $\mathfrak{p}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-2s} as v runs through the finite number of places in S_{fin}. Also if $\Psi\in\mathcal{M}(\Omega)$ and $\eta\in F_S^\times$ denote

$$\tilde{\Psi}_\eta(c)=(\eta,c)\,\Psi(c^{-1}\eta^{-1}). \tag{17}$$

One may easily check that Ψ_η is in $\mathcal{M}(\Omega)$.

Theorem 1 *Let $\Psi \in \mathcal{M}(\Omega)$, and let $\alpha \in \mathfrak{o}_S$. Then $\mathcal{D}^*(s, \Psi, \alpha)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2n}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta(s)$ depending only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}^*(s, \Psi, \alpha) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} \mathbb{N}(\alpha)^{1-2s} P_{\alpha\eta}(s)\, \mathcal{D}^*(1-s, \tilde{\Psi}_\eta, \alpha). \tag{18}$$

This is proved in Brubaker and Bump [4]; very similar results are in Eckhardt and Patterson [13].

Let $M(\Omega^2)$ be the finite-dimensional vector space of functions $\Psi : F_{\text{fin}} \times F_{\text{fin}} \longrightarrow \mathbb{C}$ such that when ε_1 and ε_2 are in Ω

$$\Psi(\varepsilon_1 c_1, \varepsilon_2 c_2) = (\varepsilon_1, c_1)\,(\varepsilon_1^{-1}\varepsilon_2, c_2)\,\Psi(c_1, c_2). \tag{19}$$

The dimension of $\mathcal{M}(\Omega^2)$ is the square of the cardinality of $F_S^\times/\Omega$, by adapting the proof of Brubaker and Bump [4], Lemma 3.

We will define a function $H(c_1, c_2)$ on $\mathfrak{o}_S^\times \times \mathfrak{o}_S^\times$ which satisfies the "twisted multiplicativity"

$$H(c_1c_1', c_2c_2') = H(c_1, c_2)H(c_1', c_2') \left(\frac{c_1}{c_1'}\right) \left(\frac{c_1'}{c_1}\right) \left(\frac{c_2}{c_2'}\right) \left(\frac{c_2'}{c_2}\right) \left(\frac{c_1}{c_2'}\right)^{-1} \left(\frac{c_1'}{c_2}\right)^{-1}. \tag{20}$$

It is understood that $H(\varepsilon_1, \varepsilon_2) = 1$ when ε_1 and ε_2 are units, so using the special case

$$\left(\frac{\varepsilon}{c}\right) = (c, \varepsilon) \text{ when } \varepsilon \in \mathfrak{o}_S^\times$$

of the reciprocity law, (20) includes the rule

$$H(\varepsilon_1 c_1, \varepsilon_2 c_2) = H(c_1, c_2)(c_1, \varepsilon_1)(c_2, \varepsilon_2)(c_2, \varepsilon_1)^{-1}. \tag{21}$$

This means that if $\Psi \in \mathcal{M}(\Omega^2)$ the function $H(c_1, c_2)\,\Psi(c_1, c_2)$ depends only on the values of c_1 and c_2 in $\mathfrak{o}_S/\mathfrak{o}_S^\times$, and so the multiple Dirichlet series Z_Ψ defined by (9) can be written down.

Specification of a function H satisfying (20) is clearly reduced to the specification of $H(p^{k_1}, p^{k_2})$ for primes p of $\mathfrak{o}_S$, and these are specified by (13).

Remark 1 It may be checked using (21) and (15) that if we change p to $q = \varepsilon p$, where ε is a unit, then this rule is unchanged, namely

$$\begin{aligned} H(1,1) &= 1, \\ H(1,q) = H(q,1) &= g(1,q), \\ H(q,q^2) = H(q^2,q) &= g(1,q)\, g(q,q^2), \\ H(q^2,q^2) &= g(1,q)^2\, g(q,q^2). \end{aligned}$$

No Hilbert symbols appear in these formulae! Thus it does not matter what representatives we chose for the prime ideals – the definition of H is invariant. This observation explains the paradox noted at the beginning of Section 2.

Now let $\mathcal{A}$ be the ring of (Dirichlet) polynomials in $q_v^{\pm 2s_1}, q_v^{\pm 2s_2}$ where v runs through the finite set S_{fin} of places. Let $\mathfrak{M} = \mathcal{A} \otimes \mathcal{M}(\Omega^2)$. We may regard elements of $\mathfrak{M}$ as functions $\Psi : \mathbb{C}^2 \times (F_S^\times)^2 \longrightarrow \mathbb{C}$ such that for all $(s_1, s_2) \in \mathbb{C}^2$ the function

$$(c_1, c_2) \longmapsto \Psi(s_1, s_2, c_1, c_2)$$

is in $\mathcal{M}(\Omega^2)$, while for all $(c_1, c_2) \in (F_S^\times)^2$, the function

$$(s_1, s_2) \longmapsto \Psi(s_1, s_2, c_1, c_2)$$

is in $\mathcal{A}$. As a notational point, we will sometimes use the notation

$$\Psi_s(c_1, c_2) = \Psi(s_1, s_2, c_1, c_2), \qquad s = (s_1, s_2) \in \mathbb{C}^2. \tag{22}$$

We identify $M(\Omega)$ with its image $1 \otimes M(\Omega)$ in $\mathfrak{M}$; this just consists of the Ψ_s that are independent of $s \in \mathbb{C}^2$.

We define two operators σ_1 and σ_2 on $\mathbb{C}^2$ by (6) and (7). They satisfy the *braid relation*

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \tag{23}$$

as well as

$$\sigma_1^2 = 1, \qquad \sigma_2^2 = 1. \tag{24}$$

The relations (23) and (24) are a presentation of the symmetric group S_3. We will denote this transformation group of $\mathbb{C}^2$ by W.

We define operators σ_1 and σ_2 on $\mathfrak{M}$ by

$$(\sigma_1\Psi_s)(c_1, c_2) = (\sigma_1\Psi)(s_1, s_2, c_1, c_2) =$$
$$\sum_{\eta \in F_S^\times/(F_S^\times)^n} (\eta, c_1c_2^{-1})\, P_{c_2c_1^{-2}\eta}(s_1)\Psi\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, \eta^{-1}c_1, c_2\right)$$

and

$$(\sigma_2\Psi_s)(c_1, c_2) = (\sigma_2\Psi)(s_1, s_2, c_1, c_2) =$$
$$\sum_{\eta \in F_S^\times/(F_S^\times)^n} (\eta, c_2)\, P_{c_1c_2^{-2}\eta}(s_2)\Psi\left(s_1 + s_2 - \frac{1}{2}, 1 - s_2, c_1, \eta^{-1}c_2\right).$$

Proposition 1 *If $\Psi \in \mathfrak{M}$, then $\sigma_1\Psi$ and $\sigma_2\Psi$ are in $\mathfrak{M}$.*

Proof Let ε_1 and $\varepsilon_2 \in \Omega$. We have

$$(\sigma_1\Psi)(s_1, s_2, \varepsilon_1 c_1, \varepsilon_2 c_2) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} (\eta, \varepsilon_1\varepsilon_2^{-1} c_1 c_2^{-1})\, P_{\varepsilon_2\varepsilon_1^{-2} c_2 c_1^{-2}\eta}(s_1)\Psi\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, \eta^{-1}\varepsilon_1 c_1, \varepsilon_2 c_2\right).$$

Making the variable change $\eta \longmapsto \varepsilon_2^{-1}\varepsilon_1^2\eta$ and using the fact that $\Psi \in \mathfrak{M}$, this equals

$$\sum_{\eta \in F_S^\times/(F_S^\times)^n} (\varepsilon_2^{-1}\varepsilon_1^2\eta, \varepsilon_1\varepsilon_2^{-1} c_1 c_2^{-1})\, P_{c_2 c_1^{-2}\eta}(s_1)\Psi\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, \eta^{-1}\varepsilon_2\varepsilon_1^{-1} c_1, \varepsilon_2 c_2\right) =$$

$$(\varepsilon_2^{-1}\varepsilon_1^2, c_1 c_2^{-1}) \sum_{\eta \in F_S^\times/(F_S^\times)^n} (\eta, \varepsilon_1\varepsilon_2^{-1} c_1 c_2^{-1})(\varepsilon_2\varepsilon_1^{-1}, \eta^{-1} c_1)(\varepsilon_1, c_2) P_{c_2 c_1^{-2}\eta}(s_1)$$

$$\times\Psi\left(1 - s_1, s_1 + s_2 - \frac{1}{2}, \eta^{-1} c_1, c_2\right) =$$

$$(\varepsilon_1, c_1)(\varepsilon_2\varepsilon_1^{-1}, c_2)(\sigma_1\Psi)(s_1, s_2, c_1, c_2),$$

proving that $\sigma_1\Psi \in \mathfrak{M}$; the case of $\sigma_2\Psi$ is similar. $\square$

Let $\mathfrak{W}$ be the group of automorphisms of $\mathfrak{M}$ generated by σ_1 and σ_2. This will turn out to be the group of functional equations of the multiple Dirichlet series. Clearly there is a homomorphism $\mathfrak{W} \longrightarrow W$, where we recall that $W \cong S_3$ is the group of transformations of $\mathbb{C}^2$ generated by σ_1 and σ_2. It is an interesting question to determine the kernel of this homomorphism $\mathfrak{W} \longrightarrow W$. One might hope that this kernel is finite and perhaps trivial if Ω.

Theorem 2 *Let $\Psi_s \in \mathfrak{M}$. The function $Z_{\Psi_s}(s_1, s_2)$ defined by (9) is convergent in the region Λ_0 defined by $\mathrm{re}(s_1), \mathrm{re}(s_2) > \frac{3}{4}$. It has meromorphic continuation to all s_1 and s_2; it is analytic except where s_1, s_2 or $\frac{3}{2} - s_1 - s_2$ equals $\frac{1}{2} \pm \frac{1}{2n}$, and it satisfies*

$$Z^*_{\sigma\Psi_s}(\sigma s) = Z^*_{\Psi_s}(s) \tag{25}$$

for all $\sigma \in \mathfrak{W}$.

This is a special case of the Theorem 5.9 in [5].

Proof The function Ψ is bounded as a function of c_1 and c_2 because Ω^2 has finite index in $(F_S^\times)^2$, by (19). To prove convergence on Λ_0 it is sufficient to show that (with $\mathrm{re}(s_1), \mathrm{re}(s_2) \geqslant \frac{3}{4} + \varepsilon$)

$$\begin{aligned} \infty &> \sum |H(c_1, c_2)\, \mathbb{N}(c_1)^{-2s_1} \mathbb{N}(c_2)^{-2s_2}| \\ &> \prod_p \sum_{k_1, k_2} |H(p^{k_1}, p^{k_2})\, |\mathbb{N}(p)^{-\frac{3}{2}(k_1+k_2)-4\varepsilon}. \end{aligned}$$

It is easy to check that for the 5 possible $(k_1, k_2) \neq (0, 0)$ such that $H(p^{k_1}, p^{k_2}) \neq 0$ we have

$$H(p^{k_1}, p^{k_2}) \mathbb{N}(p)^{-\frac{3}{2}(k_1+k_2)-4\varepsilon} = O(\mathbb{N}p^{-1-4\varepsilon}).$$

Thus

$$\sum_{k_1, k_2} |H(p^{k_1}, p^{k_2})\, |\mathbb{N}(p)^{-\frac{3}{2}(k_1+k_2)-4\varepsilon} = 1 + O(\mathbb{N}p^{-1-4\varepsilon})$$

and the convergence follows by comparison with the Dedekind zeta function.

Using standard bounds for the Gauss sums, we have

$$|H(c_1, c_2)\, \mathbb{N}(c_1)^{-2s_1} \mathbb{N}(c_2)^{-2s_2}| < \mathbb{N}(c_1)^{\frac{1}{2} - 2\,\mathrm{re}(s_1)} \mathbb{N}(c_2)^{\frac{1}{2} - 2\,\mathrm{re}(s_2)}.$$

It follows that (9) is convergent in Λ_0.

If $(c_1, c_2) = (\gamma_1 p^{k_1}, \gamma_2 p^{k_2})$ where $p \nmid \gamma_i$ and $k_1 \leqslant 2$ we say c_2 is c_1*-reduced at* p if (k_1, k_2) occur in the following table:

k_1	k_2
0	0
1	0
2	1

We say that c_2 is c_1*-reduced* if it is c_1-reduced at p for all p.

Lemma 1 *We have $H(c_1, c_2) = 0$ unless c_1 is cubefree and c_2 is a multiple of a c_1-reduced integer. If c_1 is cubefree and c_2 is a c_1-reduced integer, then $H(c_1, c_2) \neq 0$.*

Proof This is clear from the definition of H. □

Lemma 2 *Suppose that c_1 is cube-free and that c_2 is c_1-reduced. Then $c_0 = c_1 c_2^{-2} \in \mathfrak{o}_S$ and for every $\alpha \in \mathfrak{o}_S$ we have*

$$\frac{H(c_1, \alpha c_2)}{H(c_1, c_2)}(\alpha, c_2) = g(c_0, \alpha). \tag{26}$$

Proof It is clear from the definition of c_1-reduced that c_2^2 divides c_1. Since c_1 and c_2 are fixed, let $h(\alpha)$ denote the expression on the left-hand side of (26). We first check that the multiplicativity of h matches that of a Gauss sum. Let $(\alpha, \beta) = 1$. Factor $c_1 = \gamma_1\gamma_1'$ and $c_2 = \gamma_2\gamma_2'$ where $(\alpha\gamma_1\gamma_2, \beta\gamma_1'\gamma_2') = 1$. Then expanding

$$\begin{aligned}
\frac{h(\alpha\beta)}{h(\alpha)h(\beta)} &= \frac{H(\gamma_1\gamma_1', \alpha\beta\gamma_2\gamma_2')H(\gamma_1\gamma_1', \gamma_2\gamma_2')}{H(\gamma_1\gamma_1', \alpha\gamma_2\gamma_2')H(\gamma_1\gamma_1', \beta\gamma_2\gamma_2')}\frac{(\alpha\beta, c_2)}{(\alpha, c_2)(\beta, c_2)} \\
&= \left(\frac{\gamma_1}{\gamma_1'}\right)\left(\frac{\gamma_1'}{\gamma_1}\right)\left(\frac{\gamma_2}{\gamma_2'}\right)\left(\frac{\gamma_2'}{\gamma_2}\right)\left(\frac{\gamma_1}{\gamma_2'}\right)^{-1}\left(\frac{\gamma_1'}{\gamma_2}\right)^{-1} \\
&\times \left(\frac{\gamma_1}{\gamma_1'}\right)\left(\frac{\gamma_1'}{\gamma_1}\right)\left(\frac{\alpha\gamma_2}{\beta\gamma_2'}\right)\left(\frac{\beta\gamma_2'}{\alpha\gamma_2}\right)\left(\frac{\gamma_1}{\beta\gamma_2'}\right)^{-1}\left(\frac{\gamma_1'}{\alpha\gamma_2}\right)^{-1} \\
&\div \left(\frac{\gamma_1}{\gamma_1'}\right)\left(\frac{\gamma_1'}{\gamma_1}\right)\left(\frac{\gamma_2}{\beta\gamma_2'}\right)\left(\frac{\beta\gamma_2'}{\gamma_2}\right)\left(\frac{\gamma_1}{\beta\gamma_2'}\right)^{-1}\left(\frac{\gamma_1'}{\gamma_2}\right)^{-1} \\
&\div \left(\frac{\gamma_1}{\gamma_1'}\right)\left(\frac{\gamma_1'}{\gamma_1}\right)\left(\frac{\alpha\gamma_2}{\gamma_2'}\right)\left(\frac{\gamma_2'}{\alpha\gamma_2}\right)\left(\frac{\gamma_1}{\gamma_2'}\right)^{-1}\left(\frac{\gamma_1'}{\alpha\gamma_2}\right)^{-1} \\
&= \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right).
\end{aligned}$$

Comparing with (11), it is sufficient to check (26) when $\alpha = p^r$. We factor $c_1 = \gamma_1 p^{k_1}$, $c_2 = \gamma_2 p^{k_2}$ where $p \nmid \gamma_i$. Then $h(p^r)$ equals

$$\begin{aligned}
&\frac{H(\gamma_1 p^{k_1}, \gamma_2 p^{k_2+r})}{H(\gamma_1 p^{k_1}, \gamma_2 p^{k_2})}(p^r, c_2) \\
= \;& (p^r, c_2)\left(\frac{\gamma_1}{p^{k_1}}\right)\left(\frac{p^{k_1}}{\gamma_1}\right)\left(\frac{\gamma_2}{p^{k_2+r}}\right)\left(\frac{p^{k_2+r}}{\gamma_2}\right)\left(\frac{\gamma_1}{p^{k_2+r}}\right)^{-1}\left(\frac{p^{k_1}}{\gamma_2}\right)^{-1} H(p^{k_1}, p^{k_2+r})H(\gamma_1, \gamma_2) \\
\div \;& \left[\left(\frac{\gamma_1}{p^{k_1}}\right)\left(\frac{p^{k_1}}{\gamma_1}\right)\left(\frac{\gamma_2}{p^{k_2}}\right)\left(\frac{p^{k_2}}{\gamma_2}\right)\left(\frac{\gamma_1}{p^{k_2}}\right)^{-1}\left(\frac{p^{k_1}}{\gamma_2}\right)^{-1} H(p^{k_1}, p^{k_2})H(\gamma_1, \gamma_2)\right] \\
= \;& (p^r, c_2)\left(\frac{\gamma_2}{p^r}\right)\left(\frac{p^r}{\gamma_2}\right)\left(\frac{\gamma_1}{p^r}\right)^{-1} \frac{H(p^{k_1}, p^{k_2+r})}{H(p^{k_1}, p^{k_2})}.
\end{aligned}$$

Since $(p, p) = 1$, $(p^r, c_2) = (p^r, \gamma_2)$, so by the reciprocity law

$$h(p^r) = \left(\frac{\gamma_2^{-2}\gamma_1}{p^r}\right)^{-1} \frac{H(p^{k_1}, p^{k_2+r})}{H(p^{k_1}, p^{k_2})}.$$

Table (13) shows that for k_1 and k_2 given (such that p^{k_2} is p^{k_1}-reduced) there are exactly two values of r for which $h(p^r)$ is nonzero: they are $r = 0$ and $r = k_1 - 2k_2 + 1 = \mathrm{ord}_p(c_0) + 1$.

If $r=0$, then $h(1)=1$. We may therefore assume that $r=\mathrm{ord}_p(c_0)+1$. In this case, we have

$$\frac{H(p^{k_1},p^{k_2+r})}{H(p^{k_1},p^{k_2})}=g(p^{r-1},p^r)$$

as may be seen from the following table.

k_1	k_2	$r=\mathrm{ord}_p(c_0)+1$	$H(p^{k_1},p^{k_2+r})/H(p^{k_1},p^{k_2})$
0	0	1	$g(1,p)$
1	0	2	$g(p,p^2)$
2	1	1	$g(1,p)$

We have

$$h(p^r)=\left(\frac{\gamma_2^{-2}\gamma_1}{p^r}\right)^{-1}g(p^{r-1},p^r)=g(\gamma_1\gamma_2^{-2}p^{r-1},p^r).$$

We are done since with our assumption that $r=\mathrm{ord}_p(c_0)+1$ we have $\gamma_1\gamma_2^{-2}p^{r-1}=c_0$. This proves the Lemma. □

For c_1 and c_2 given, if $\Psi\in\mathcal{M}(\Omega^2)$, or more generally if $\Psi=\Psi_s\in\mathfrak{M}$ let

$$\Psi^{c_1,c_2}(\alpha)=\Psi(c_1,\alpha c_2)\,(\alpha,c_2)^{-1}.$$

Lemma 3 *The function $\Psi^{c_1,c_2}\in\mathcal{M}(\Omega)$.*

Proof This is easily checked using (19). □

Lemma 4 *We have*

$$Z_\Psi(s_1,s_2)=\sum_{\substack{0\neq c_1,c_2\in\mathfrak{o}_S^\times\backslash\mathfrak{o}_S\\ c_1\ \textit{cube-free}\\ c_2\ c_1\textit{-reduced}}}\mathbb{N}c_1^{-2s_1}\,\mathbb{N}c_2^{-2s_2}H(c_1,c_2)\,\mathcal{D}(s_2,\Psi^{c_1,c_2},c_1c_2^{-2}).\tag{27}$$

Proof By Lemma 1, we may rewrite (9) by first summing over c_1 cubefree, then replacing c_2 by c_2d_2, where c_2 is a fixed generator of the ideal of c_1-reduced elements of $\mathfrak{o}_S$ and d_2 is summed over $\mathfrak{o}_S/\mathfrak{o}_S^\times$. Then invoking (26) the summation over d_2 produces $\mathcal{D}(s_2,\Psi^{c_1,c_2},c_1c_2^{-2})$, and the statement follows. □

Lemma 5 *We have, for $\Psi\in\mathfrak{M}$*

$$\mathcal{D}^*(s_2,\Psi^{c_1,c_2},c_1c_2^{-2})=\mathbb{N}(c_1c_2^{-2})^{1-2s_2}\mathcal{D}^*(1-s_2,(\sigma_2\Psi)^{c_1,c_2},c_1c_2^{-2}).\tag{28}$$

Proof By Theorem 1 we have

$$\mathcal{D}^*(s_2, \Psi^{c_1,c_2}, c_1c_2^{-2}) = \sum_{\eta \in F_S^\times/(F_S^\times)^n} \mathbb{N}(c_1c_2^{-2})^{1-2s_2} P_{c_1c_2^{-2}\eta}(s_2)\mathcal{D}^*(1-s_2, \tilde{\Psi}_\eta^{c_1,c_2}, c_1c_2^{-2}),$$

where with our definitions

$$\tilde{\Psi}_\eta^{c_1,c_2}(d) = (\eta, d)\Psi^{c_1,c_2}(d^{-1}\eta^{-1}) = (\eta, d)\,(d\eta, c_2)\,\Psi(c_1, d^{-1}\eta^{-1}c_2).$$

(We are suppressing the dependence of Ψ on s from the notation.) Now we take s_2 to have large negative real part so that the right-hand side can be expanded out as a Dirichlet series. Thus $\mathcal{D}^*(s_2, \Psi^{c_1,c_2}, c_1c_2^{-2})$ equals

$$\begin{aligned}
\sum_{\eta \in F_S^\times/(F_S^\times)^n} \mathbb{N}(c_1c_2^{-2})^{1-2s_2} P_{c_1c_2^{-2}\eta}(s_2)\mathcal{D}^*(1-s_2, \tilde{\Psi}_\eta^{c_1,c_2}, c_1c_2^{-2}) &= \\
\sum_{\eta \in F_S^\times/(F_S^\times)^n} \mathbb{N}(c_1c_2^{-2})^{1-2s_2} P_{c_1c_2^{-2}\eta}(s_2) & \\
\sum_{0\neq d \in \mathfrak{o}_S/\mathfrak{o}_S^\times} g(c_1c_2^{-2}, d)\,\mathbb{N}d^{2s_2-2}(\eta, d\,c_2)(d, c_2)\,\Psi(c_1, d^{-1}\eta^{-1}c_2) &= \\
\sum_{0\neq d \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \sum_{\eta \in F_S^\times/(F_S^\times)^n} \mathbb{N}(c_1c_2^{-2})^{1-2s_2} P_{c_1c_2^{-2}d^{-2}\eta}(s_2) & \\
g(c_1c_2^{-2}, d)\,\mathbb{N}d^{2s_2-2}(d^{-2}\eta, d\,c_2)(d, c_2)\,\Psi(c_1, d\eta^{-1}c_2) &= \\
\sum_{0\neq d \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \sum_{\eta \in F_S^\times/(F_S^\times)^n} \mathbb{N}(c_1c_2^{-2})^{1-2s_2} P_{c_1c_2^{-2}d^{-2}\eta}(s_2) & \\
g(c_1c_2^{-2}, d)\,\mathbb{N}d^{2s_2-2}(\eta, d\,c_2)(d, c_2)^{-1}\,\Psi(c_1, \eta^{-1}dc_2) &= \\
\sum_{0\neq d \in \mathfrak{o}_S/\mathfrak{o}_S^\times} \mathbb{N}(c_1c_2^{-2})^{1-2s_2}(\sigma_2\Psi)\,(s_1, s_2, c_1, dc_2)\,g(c_1c_2^{-2}, d)\,\mathbb{N}d^{2s_2-2}(d, c_2)^{-1} &= \\
\mathbb{N}(c_1c_2^{-2})^{1-2s_2}\mathcal{D}^*(1-s_2, (\sigma_2\Psi)^{c_1,c_2}, c_1c_2^{-2}), &
\end{aligned}$$

where we have made a variable change $\eta \longmapsto d^{-2}\eta$ and used the fact that $(d, d) = 1$. This completes the proof of the Lemma. $\square$

Lemma 6 *The function $Z_\Psi^*(s_1, s_2)$ has meromorphic continuation to the region Λ_2 defined by (8), and satisfies the functional equation*

$$Z_{\sigma_2\Psi}^*(\sigma_2 s) = Z_\Psi^*(s).$$

It is analytic except where $s_2 = \frac{1}{2} \pm \frac{1}{2n}$.

Proof The expression (27) gives the continuation to the region Ω_2. The functional equation follows by combining (27) with (28); note that of the normalizing factor of Z,

$$\boldsymbol{G}_n(s_2)\,\zeta_F(2ns_2 - n + 1)$$

is needed to normalize $\mathcal{D}$, while the remaining parts are interchanged by the transformation (7). $\square$

There is of course also a second functional equation corresponding to (6). This is similar and we omit details of it. We note that the Hilbert symbols that appear look slightly different since we made an arbitrary choice in writing (20) by choosing the last two symbols to be

$$\left(\frac{c_1}{c_2'}\right)^{-1}\left(\frac{c_1'}{c_2}\right)^{-1} \quad \text{instead of} \quad \left(\frac{c_2'}{c_1}\right)^{-1}\left(\frac{c_2}{c_1'}\right)^{-1}.$$

Thus merely interchanging the two coordinates does not quite preserve the space $\mathcal{M}(\Omega^2)$. Instead, if $\Psi \in \mathcal{M}(\Omega^2)$ then so is Ψ' defined by

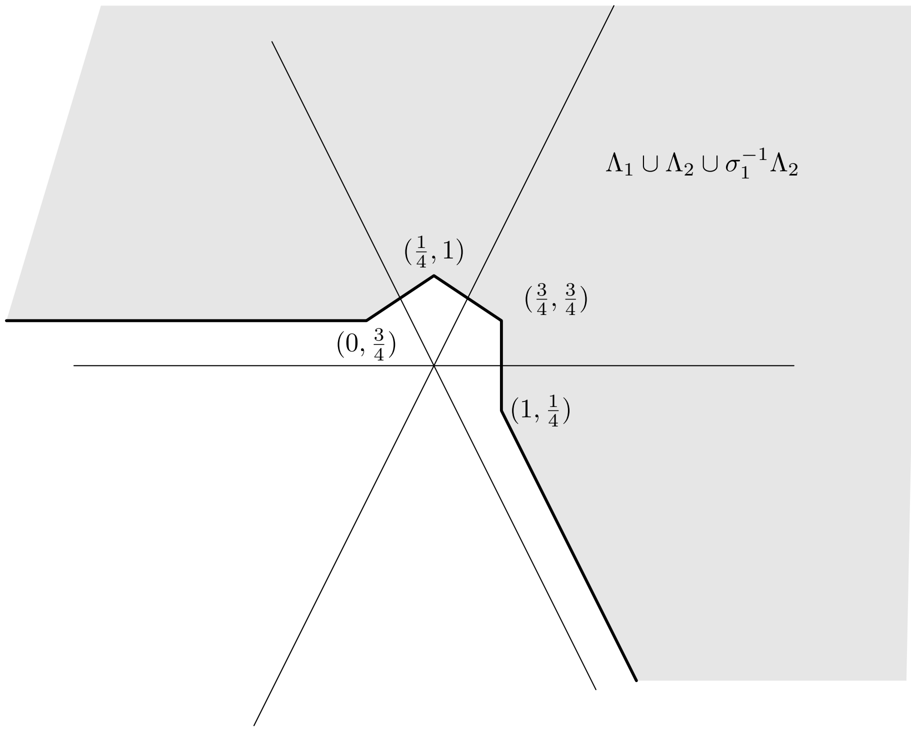
$$\Psi'(c_1, c_2) = (c_2, c_1)\Psi(c_2, c_1),$$

and conjugating σ_2 by this involution of $\mathcal{M}(\Omega^2)$ (while interchanging the roles of s_1 and s_2) gives the transformation σ_1 of $\mathfrak{M}$. We have, as in Lemma 6 the functional equation

$$Z^*_{\sigma_1\Psi}(\sigma_1 s) = Z^*_\Psi(s). \tag{29}$$

We may now prove the global meromorphic continuation of $Z^*(s)$. First we obtain continuation of Z_Ψ to the region (with Λ_1 and Λ_2 as in Section 1)

$$\Lambda_1 \cup \Lambda_2 \cup \sigma_1^{-1}\Lambda_2.$$



On Λ_2, we have already noted the analytic continuation in Lemma 6, and the continuation to Λ_1 is given by the same method, and on the region Λ_1 we have (29). Of course, the two continuations agree on the overlap $\Lambda_1 \cap \Lambda_2 = \Lambda_0$. Now the continuation to $\sigma_1\Lambda_2$ is obtained by the formula

$$Z^*_\Psi(\sigma_1^{-1}s) = Z^*_{\sigma_1\Psi}(s), \qquad s \in \Lambda_2, \tag{30}$$

and we must check that the continuations agree on the overlap $\Lambda_1 \cap \Lambda_2 \cap \sigma_1^{-1}\Lambda_2 = \sigma_1\Lambda_0$; indeed, if $s = \sigma_1 s'$ where $s' \in \Lambda_0$ then the right-hand side of (30) equals

$$Z^*_\Psi(\sigma_1^{-1}s) = Z^*_{\sigma_1\Psi}(s) = Z^*_{\sigma_1\Psi}(\sigma_1 s') = Z^*_\Psi(s'),$$

where at the last step we have used (29). This is the same as the left-hand side of (30). In conclusion, we have obtained the analytic continuation of Z_Ψ as a well-defined function on $\Lambda_1 \cup \Lambda_2 \cup \sigma_1^{-1}\Lambda_2$. This region is a simply connected tube domain whose convex hull is all of $\mathbb{C}^2$, and so the meromorphic continuation to $\mathbb{C}^2$ follows from Bochner's Tube Domain Theorem (Bochner [1] or Hormander [16], Theorem 2.5.10). Note that Bochner's theorem applies to analytic functions; so we apply it to the function

$$\left(s_1 - \frac{1}{2} - \frac{1}{2n}\right)\left(s_1 - \frac{1}{2} + \frac{1}{2n}\right)\left(s_2 - \frac{1}{2} - \frac{1}{2n}\right)\left(s_2 - \frac{1}{2} + \frac{1}{2n}\right)$$
$$\left(1 - s_1 - s_2 - \frac{1}{2n}\right)\left(1 - s_1 - s_2 + \frac{1}{2n}\right) Z_\Psi(s).$$

Now when $\sigma = \sigma_1$ or σ_2, (25) is known when $s \in \Lambda_0$, and by analytic continuation it is therefore true for all $s \in \mathbb{C}^2$. It follows that it is true for the group $\mathfrak{W}$ they generate.

This completes the proof of Theorem 2. □

4 Examples

In this section, we will discuss some examples. We return to the "heuristic" viewpoint of Section 1. The heuristic point of view is best for quickly grasping the relationships between various objects, before making those relationships rigorous along the lines of Section 3.

In the case where $n = 2$, there is little harm in replacing the quadratic Gauss sum $g(\alpha, c)$ by simply $\left(\frac{\alpha}{c}\right) \mathbb{N}c^{-1/2}$ in the definition of $Z(s)$. (For simplicity, let us restrict ourselves to the simply-laced case.) Thus we obtain the heuristic form

$$Z(s) = \sum_{\substack{0 \neq c_\alpha \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (\alpha \in \Delta)}} \left[\prod_{\alpha, \beta \text{ not orthogonal}} \left(\frac{c_\alpha}{c_\beta}\right)\right] \prod_{\alpha \in \Delta} \mathbb{N}(c_\alpha)^{\frac{1}{2} - 2s_\alpha}.$$

In the three cases where $\Phi = A_2, A_3$ and D_4, the Dynkin diagram of Φ is a "star" with one vertex adjacent to all others; let us call the corresponding root α_1, and the others $\alpha_2, \cdots, \alpha_r$, where $r = 2, 3$ or 4 is the rank. Then, denoting $D = c_{\alpha_1}$ and $c_i = c_{\alpha_i}$ $(i = 2, \cdots, r)$ we have

$$Z(s) = \sum_{\substack{D, c_i \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (\alpha \in \Delta)}} \left(\frac{c_2}{D}\right) \cdots \left(\frac{c_r}{D}\right) \mathbb{N}(D)^{\frac{1}{2} - 2s_\alpha}.$$

Denoting by χ_D the Hecke character which maps the principal ideal generated by α to $\left(\frac{D}{\alpha}\right)$, we may write this

$$Z(s) = \sum_D L\left(2s_2 - \frac{1}{2}, \chi_D\right) \cdots L\left(2s_r - \frac{1}{2}, \chi_D\right) \mathbb{N}(D)^{\frac{1}{2} - 2s_1}.$$

It is also possible to replace $L\left(2s_2 - \frac{1}{2}, \chi_D\right) \cdots L\left(2s_r - \frac{1}{2}, \chi_D\right)$ in this expression by $L(2s - \frac{1}{2}, f, \chi_D)$, where f is an automorphic form on GL_r, with $r = 1, 2$ or 3. In this case, they Weyl group $\Phi = A_2, A_3$ and D_4 is replaced by its subgroup, A_2, B_2 or G_2. These examples were considered in Bump, Friedberg and Hoffstein [10].

As we mentioned in the introduction, there is strong reason to believe that the Weyl group multiple Dirichlet series (for arbitrary n and Φ) are Whittaker coefficients of Eisenstein series on metaplectic covers of semisimple algebraic groups. (We will return to this point in a later paper.) But when $n = 2$, there may also be *nonmetaplectic* Rankin-Selberg constructions. For example, applying the construction of Maass [19] to an Eisenstein series of Klingen type and applying results of Mizumoto [20] one obtains the B_2 example; for the G_2 example, one relies on an unpublished construction of David Ginzburg involving SO_7. Another "nonmetaplectic" context in which these multiple Dirichlet series occur is the discovery by Venkatesh [23] that when $n = 2$ the A_r multiple Dirichlet series are related to periods of nonmetaplectic Eisenstein series.

Returning to general n, Friedberg, Hoffstein and Lieman [14] have considered multiple Dirichlet series with the heuristic form

$$Z(s, w) = \sum_D L\left(2s - \frac{1}{2}, \chi_D\right) \mathbb{N}(D)^{\frac{1}{2} - 2w},$$

where now $\chi_D(c) = \left(\frac{D}{c}\right)$ in terms of the n-th order symbol; these were applied to mean values of $L(s, \chi_D)$. When $n > 2$ there are actually two distinct objects which are interchanged by the functional equations, which form a nonabelian group of order 32. There is convincing reason to believe that this multiple Dirichlet series is a *residue* of the A_n multiple Dirichlet series, and this has been checked when $n = 3$ (see [3]).

Brubaker, Friedberg and Hoffstein [7] considered a multiple Dirichlet series with the heuristic form

$$\sum_D L\left(2s - \frac{1}{2}, f \otimes \chi_D\right) \mathbb{N}(D)^{\frac{1}{2} - 2w}$$

where $n = 3$ and f is an automorphic form on GL_2. The group of functional equations is of order 384. Although this is the same as the order of the classical Weyl group of type B_3, it is not the same group. Brubaker showed that the cusp form may be replaced by an Eisenstein series, in which case one has a meromorphic function of 3 variables. This appears to be a residue of the E_6 multiple Dirichlet series (with $n = 3$).

Chinta [12] made use of the A_5 multiple Dirichlet series (with $n = 2$) in order to study the distribution of central values of biquadratic zeta functions. We note that Chinta's example is outside the stable case.

Finally, we take this opportunity to point out that as noted in the introduction to [5] the residues the A_3 multiple Dirichlet series when $n = 4$ are clearly connected with the conjecture of Patterson [21].

References

[1] S. Bochner. A theorem on analytic continuation of functions in several variables. *Ann. of Math. (2)*, 39(1):14–19, 1938.

[2] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[3] B. Brubaker and D. Bump. Residues of Weyl group multiple Dirichlet series associated to $\widetilde{\mathrm{GL}}_{n+1}$.

[4] B. Brubaker and D. Bump. On Kubota's Dirichlet series. *J. Reine Angew. Math.*, to appear.

[5] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series II. The stable case. *Invent. Math., to appear.*

[6] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_r. *Preprint, available at* `http://sporadic.stanford.edu/bump/wmd3.pdf`.

[7] B. Brubaker, S. Friedberg, and J. Hoffstein. Cubic twists of GL(2) automorphic L-functions. *Invent. Math.*, 160(1):31–58, 2005.

[8] D. Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2004.

[9] D. Bump and S. Friedberg. The exterior square automorphic L-functions on GL(n). In *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, volume 3 of *Israel Math. Conf. Proc.*, pages 47–65. Weizmann, Jerusalem, 1990.

[10] D. Bump, S. Friedberg, and J. Hoffstein. On some applications of automorphic forms to number theory. *Bull. Amer. Math. Soc. (N.S.)*, 33(2):157–175, 1996.

[11] D. Bump and J. Hoffstein. On Shimura's correspondence. *Duke Math. J.*, 55(3):661–691, 1987.

[12] G. Chinta. Mean values of biquadratic zeta functions. *Invent. Math.*, 160(1):145–163, 2005.

[13] C. Eckhardt and S. J. Patterson. On the Fourier coefficients of biquadratic theta series. *Proc. London Math. Soc. (3)*, 64(2):225–264, 1992.

[14] S. Friedberg, J. Hoffstein, and D. Lieman. Double Dirichlet series and the n-th order twists of Hecke L-series. *Math. Ann.*, 327(2):315–338, 2003.

[15] Jeff Hoffstein. Eisenstein series and theta functions on the metaplectic group. In *Theta functions: from the classical to the modern*, volume 1 of *CRM Proc. Lecture Notes*, pages 65–104. Amer. Math. Soc., Providence, RI, 1993.

[16] L. Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[17] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[18] T. Kubota. *On automorphic functions and the reciprocity law in a number field.* Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co. Ltd., Tokyo, 1969.

[19] H. Maass. *Siegel's modular forms and Dirichlet series.* Springer-Verlag, Berlin, 1971. Dedicated to the last great representative of a passing epoch. Carl Ludwig Siegel on the occasion of his seventy-fifth birthday, Lecture Notes in Mathematics, Vol. 216.

[20] S. Mizumoto. Fourier coefficients of generalized Eisenstein series of degree two. I. *Invent. Math.*, 65(1):115–135, 1981/82.

[21] S. J. Patterson. Whittaker models of generalized theta series. In *Seminar on number theory, Paris 1982–83 (Paris, 1982/1983)*, volume 51 of *Progr. Math.*, pages 199–232. Birkhäuser Boston, Boston, MA, 1984.

[22] A. Selberg. A new type of zeta functions connected with quadratic forms. In *Report of the Institute in the Theory of Numbers*, pages 207–210. University of Colorado, Boulder, Colorado, 1959.

[23] A. Venkatesh. Private communication.