UNITARY PERIODS, HERMITIAN FORMS AND POINTS ON FLAG VARIETIES

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Abstract. Let E be an imaginary quadratic extension of $\mathbb{Q}$ of class number one. We examine certain representation numbers associated to Hermitian forms over E, which involve counting integral points on flag varieties.

1. Introduction

The study of representation numbers of integral quadratic and Hermitian forms is a topic of classical interest. For example, an identity of Jacobi says that the number of ways to write a positive integer n as a sum of four integer squares is equal to $8 \sum d$ where the sum is over all divisors of n which are not divisible by 4. This result and many related results on representation numbers of quaternary quadratic forms were reinterpreted by Elstrodt, Grunewald and Mennicke [EGM87] as results about representation numbers of binary Hermitian forms over an imaginary quadratic number field E with ring of integers $\mathcal{O}$. They then related these to weighted sums of point evaluations of Eisenstein series for the group $PSL_2(\mathcal{O})$ acting on hyperbolic 3-space. This weighted sum can be interpreted adèlically as a period integral of the Eisenstein series over a unitary group.

More recently, formulas for unitary periods of Eisenstein series for the group $GL_n(E)$ have been obtained by Lapid-Rogawski [LR00] (for $n = 3$) and [Off] (for general n). As in the work of [EGM87], these formulas equate the period integral with a finite sum of Euler products. We remark however that the local terms in [EGM87] are local densities that they compute explicitly at all places. Formulas for the same local densities were obtained by Y. Hironaka [Hir89]. Hironaka generalized the computation of local densities in a series of papers [Hir88b, Hir99, Hir98] and finally obtained a general formula for local densities of Hermitian forms in [Hir00]. Though explicit, the formula is rather complicated. In [Hir88a], Hironaka introduced spherical functions on the space of Hermitian matrices associated to a quadratic

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extension of p-adic fields. She obtained a formula relating the spherical functions to local densities [Hir88a, §2 Theorem]. Although the formula indicates a strong relation between spherical functions and local densities, it is not yet clear in general how explicit formulas for the latter can provide explicit formulas for the first. The local data that appears in the formula of [LR00, Off] for the unitary period of an Eisenstein series is in terms of Hironaka's spherical functions, explicit formulas for which are available in [Hir99, Theorem 1] for the case of an unramifed quadratic extension. For the case of a ramified extension, explicit formulas are only available if $n = 2$. Thus, in contrast to [EGM87], the local terms in the results for $n > 2$ [LR00, Off] are explicit only outside a finite set of primes.

The purpose of the current work is to give an arithmetic application of the formula for the unitary period. For simplicity, we restrict our attention to an imaginary quadratic field E of class number one. We express the unitary period of an Eisenstein series induced from a standard parabolic subgroup P of $G = GL_n$ as a Dirichlet series whose coefficients are certain representation numbers related to counting points on the (partial) flag variety $P\backslash G$. Special cases reduce to more familiar representation numbers. For example, generalizing the setting of [EGM87], consider the Eisenstein series E_P associated to the parabolic P of type $(n-1,1)$ of G. Let $\mathcal{O}^n_{prim}$ be the set of column vectors ${}^t(v_1,\dots,v_n) \in \mathcal{O}^n$ such that the ideal generated by the v_i's is $\mathcal{O}$. Let $g \in GL_n(\mathbb{C})$ be such that the associated positive definite Hermitian form

$$Q : v \mapsto {}^t\bar{v}g{}^t\bar{g}v$$

is integral. The Eisenstein series $E_P(g;\cdot)$ can be expressed as a Dirichlet series whose m-th coefficient is

$$\#\{v \in \mathcal{O}^n_{prim} : Q(v) = m\},$$

the number of ways to represent m by the Hermitian form Q with primitive integral vectors.

For a second example, let P be the parabolic of type $(1, n-2, 1)$. Then $E_P(g;\cdot)$ is a Dirichlet series in two complex variables whose (m_1,m_2) coefficient is

$$\#\{v \in \mathcal{O}^n_{prim},\ w \in |\det g|^2\, g^{-1}({}^t\bar{g})^{-1}\mathcal{O}^n_{prim} : \\ Q(v) = m_1,\ Q(w) = |\det g|^2\, m_2,\ Q(v,w) = 0\} \tag{1.1}$$

where $Q(v,w) = {}^t\bar{v}g{}^t\bar{g}w$. In particular, if $g = e$ is the identity matrix, then this is the number of ways to represent the diagonal matrix $\mathrm{diag}(m_1,m_2)$ by Q with a $2 \times n$ integral matrix with primitive rows.

There exists a very general theory of representation numbers of one form by another, developed by Siegel for quadratic forms [Sie35, Sie36, Sie37] and extended to Hermitian forms by H. Braun [Bra41]. For more information, see the recent survey of Schulze-Pillot [SP04]. The representation numbers that arise from our formulas for parabolics other than those described in the above two examples, however, are not of the form considered by Siegel and Braun. For an example, take $n \geq 4$ with $P = B$ the standard Borel subgroup and U its unipotent radical. We have the "Plücker embedding"

$$U(\mathcal{O})\backslash SL_n(\mathcal{O}) \hookrightarrow \prod_{i=1}^{n-1} \mathcal{O}^{\binom{n}{i}} \tag{1.2}$$

$$h \mapsto (v_1(h), \ldots, v_{n-1}(h)) \tag{1.3}$$

where $v_i(h) \in \mathcal{O}^{\binom{n}{i}}$ is the vector of $i \times i$ minors in the bottom i rows of h. Let $\mathcal{I} \subset \prod_{i=1}^{n-1} \mathcal{O}^{\binom{n}{i}}$ be the image of this embedding. We define

$$r_B(Q; k_1, \ldots, k_{n-1}) = \\ \#\{(v_1, \ldots, v_{n-1}) \in \mathcal{I} : Q_i(v_i) = k_{n-i},\ i = 1, \ldots, n-1\}$$

where Q_i is the Hermitian form on $\mathbb{C}^{\binom{n}{i}}$ associated to $\wedge^i(g^t \bar{g})$. This representation number is a coefficient of the Dirichlet series representing the value at g of the Eisenstein series induced from the Borel. Computing a unitary period of this Eisenstein series amounts to computing the weighted sum

$$\sum_{Q'} \frac{1}{\epsilon(Q')} \sum_{k_1, \ldots, k_{n-1} \geq 1} \frac{r_B(Q'; k_1, \ldots, k_{n-1})}{k_1^{s_1} \ldots k_{n-1}^{s_{n-1}}} \tag{1.4}$$

where the sum is over classes in the genus class of Q and $\epsilon(Q)$ is the size of the group of integral isometries preserving Q. Our main result implies, in particular, the following.

Theorem 1.1. *Let g and Q be as above. Let $x = g^t \bar{g}$ and assume that x is in the $G(\mathcal{O}_{v_0})$-orbit of the identity for v_0 the place of E dividing the discriminant Δ_E of E. We then have*

$$\sum_{Q'} \frac{1}{\epsilon(Q')} \sum_{(k_1 \cdots k_{n-1}, l)=1} \frac{r_B(Q'; k_1, \ldots, k_{n-1})}{k_1^{\lambda_1 - \lambda_2 + 1} \ldots k_{n-1}^{\lambda_{n-1} - \lambda_n + 1}} = \\ w_E^{-1} \det x^{-(\lambda_1 + \frac{n-1}{2})} \prod_{p \nmid \Delta_E} P_{m(x_p)}(\lambda) \left(\prod_{i<j} \frac{L_p(\eta^{i+j+1}, \lambda_i - \lambda_j)}{L_p(\eta^{i+j}, \lambda_i - \lambda_j + 1)} \right).$$

Here, w_E is the number of units in $\mathcal{O}$, η is the quadratic Dirichlet character associated to $E/\mathbb{Q}$ and $L_p(\eta^i, s) = (1-\eta^i(p)p^{-s})^{-1}$ for $p \nmid \Delta_E$ is the local Euler factor of the L-function $L(\eta^i, s)$. The expression $P_{m(x_p)}(\lambda)$ is a polynomial in $p^{\lambda_1}, \dots, p^{\lambda_n}$ given explicitly in (3.11).

Remark 1. This theorem is Corollary 3.1 applied to the minimal parabolic B of G.

Remark 2. As we assume class number one, there is a unique prime $l \mid \Delta_E$ and therefore v_0 is well defined.

Remark 3. In some cases of small rank, there is a unique class in the genus class of the Hermitian form associated with the identity matrix. For example, in [Fei78] W. Feit classifies all unimodular lattices over $\mathbb{Z}[\omega]$ of rank at most 12, where ω is a cube root of -1. Over $\mathbb{Z}[i]$ similar results were obtained by Iyanaga [Iya69]. A. Schiemann has computed [Sch98] more extensive tables of class numbers of positive definite unimodular Hermitian forms over the ring of integers of more general imaginary quadratic fields. These are available at the web page `http://www.math.uni-sb.de/ag/schulze/Hermitian-lattices`. We make use of the results of Feit and Iyanaga in §4, where we give examples of the representation numbers of a single Hermitian form in some special cases.

Remark 4. The expression $P_{m(x_p)}(\lambda)$ equals one whenever x_p is in the K_p-orbit of the identity, where $K_p = GL_n(\mathbb{Z}_p) \times GL_n(\mathbb{Z}_p)$ if p is split and $K_p = GL_n(\mathcal{O}_v)$ if p is inert and v is the place of E above p. Consequently, the product over primes appearing in the theorem is essentially a quotient of products of Dirichlet L-functions.

We fix here some notation regarding L-functions. First, $\zeta_E(s)$ is the Dedekind zeta function of E and $\zeta = \zeta_{\mathbb{Q}}$. We let $(\zeta_E)_{-1} = \mathrm{Res}_{s=1}(\zeta_E(s))$. For a Dirichlet character χ we let $L(\chi, s) = \prod_p L_p(\chi, s)$ be the (finite part of) the Dirichlet L-function. If $L(s)$ is either a Dirichlet L-function or a Dedekind zeta function we denote by $L^*(s)$ the completed L-function (including the archimedean factors) and by $L^{(D)}(s)$ the partial L-function away from primes dividing the integer D.

2. An anisotropic unitary period as a finite sum over a genus class

For a number field F, we denote by $\mathbb{A}_F$ the ring of adèles of F and by $\mathbb{A}_{F,f}$ its subring of finite adèles. We will also denote $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. For an algebraic set G defined over F and a place v of F we denote $G_v = G(F_v)$ and we let $G_{\mathbb{A}_F} = G(\mathbb{A}_F)$.

Let E be an imaginary quadratic extension of $\mathbb{Q}$ of discriminant Δ_E. Trough out this work we assume that E has class number one. Denote by $\mathcal{O} = \mathcal{O}_E$ the ring of integers of E and let $w_E = \#\mathcal{O}^\times$. For any place p of $\mathbb{Q}$ we denote $E_p = E \otimes_F F_p$. Thus $E_\infty = \mathbb{C}$, $E_p = F_p \oplus F_p$ if p is split in E and $E_p/\mathbb{Q}_p$ is a quadratic extension if p is inert in E. Let G be the group GL_n regarded as an algebraic group defined over E. It will also be convenient to denote $G_\infty = GL_n(\mathbb{C})$, $G_p = GL_n(\mathbb{Q}_p) \times GL_n(\mathbb{Q}_p)$ for a split prime p and $G_p = GL_n(E_p)$ for an inert prime p. Let K be the standard maximal compact subgroup of $G_{\mathbb{A}_E}$, i.e.

$$K = U(n) \prod_{v<\infty} GL_n(\mathcal{O}_v)$$

where $U(n) = K_\infty$ is the unitary group in $GL_n(\mathbb{C})$ and the product is over all places of E. It will also be convenient to write $K = \prod_p K_p$ where for finite p we have $K_p = GL_n(\mathbb{Z}_p) \times GL_n(\mathbb{Z}_p)$ if p is split and $K_p = GL_n(\mathcal{O}_v)$ if p is inert and v is the place of E above p. For an object Y which is the restricted product $Y = \prod_p Y_p$ over all places of $\mathbb{Q}$ we will denote $Y_f = \prod_{p<\infty} Y_p$. Let

$$X = \{g \in G : {}^t\bar{g} = g\}$$

be the space of Hermitian matrices in G. There is an action of G on X given by $g \cdot x = gx^t\bar{g}$. For $x \in X$ we let

$$H^x = \{g \in G : g \cdot x = x\}$$

be the unitary group associated with x. For $x \in X_\mathbb{Q}$ we define the class of x to be

$$[x] = GL_n(\mathcal{O}) \cdot x$$

and denote $x \sim y$ if $y \in [x]$. Also define the genus class of x to be

$$[[x]] = X_\mathbb{Q} \cap (G_\infty K_f) \cdot x$$

and let $[[x]]/\sim$ be the set of classes in the genus class of x. Let X_∞^+ be the set of positive definite Hermitian matrices in X_∞. It is well known that if $x \in X_\mathbb{Q}$ is such that $x_\infty \in X_\infty^+$ then $[[x]]/\sim$ is a finite set. Let $x \in X_\mathbb{Q}$ be positive definite at infinity, and let $\theta \in G_\infty$ be such that

$$\theta \cdot e = x. \tag{2.1}$$

We denote

$$\mathcal{E}(x) = \{g \in G_\mathcal{O} : g \cdot x = x\} \text{ and } \epsilon(x) = \#\mathcal{E}(x).$$

Recall that since E is of class number one we have $G_\mathbb{A} = G_\mathbb{Q} G_\infty K_f$. It follows that the imbedding of G_∞ in $G_\mathbb{A}$ defines a bijection

$$G_E \backslash G_{\mathbb{A}_E}/K \simeq G_\mathcal{O} \backslash G_\infty / K_\infty = GL_n(\mathcal{O}) \backslash GL_n(\mathbb{C})/U(n).$$

The symmetric space $GL_n(\mathbb{C})/U(n)$ is identified with X_∞^+ via $g \mapsto g \cdot e$. Thus a function ϕ on $G_E\backslash G_{\mathbb{A}_E}/K$ can be regarded as a function ϕ^+ on $G_{\mathcal{O}}\backslash X_\infty^+$ by setting $\phi^+(g\cdot e) = \phi(g)$, $g \in G_\infty$. For the case of positive definite quadratic forms the analogue of the following lemma is proved in [Bor63]. For the convenience of the reader we repeat the proof here.

Lemma 2.1. *Let ϕ be a function on $G_E\backslash G_{\mathbb{A}_E}/K$ then for all $x \in X_\infty^+$ we have*

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}} \phi(h\theta)dh = \mathrm{vol}((H^x_{\mathbb{A}_f}\cap K_f)H^e_\infty) \sum_{[y]\in[[x]]/\sim} \epsilon(y)^{-1}\phi^+(y).$$

Proof. First we define a map

$$\mathfrak{i} : H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}/(H^x_{\mathbb{A}_f}\cap K_f)H^x_\infty \to [[x]]/\sim$$

as follows. For any $h \in H^x_{\mathbb{A}}$ we write $x = N^{-1}M$ with $N \in G_{\mathbb{Q}}$ and $M \in G_\infty K_f$. We set

$$\mathfrak{i}(h) = [N\cdot x].$$

We check that the map is well defined. If $h = N'^{-1}M'$ is a second such decomposition then $N'N^{-1} \in G_E \cap G_\infty K_f \subset G_{\mathcal{O}}$ and therefore $[N\cdot x] = [N'\cdot x]$. Note also that if $\gamma \in H^x_{\mathbb{Q}}$ and $k \in (H^x_{\mathbb{A}_f}\cap K_f)H^x_\infty$ then $\gamma h k = (N\gamma^{-1})^{-1}(Mk)$ with $N\gamma^{-1} \in G_E$ and $Mk \in G_\infty K_f$. Since $\gamma^{-1}\cdot x = x$ we see that indeed $\mathfrak{i}$ is a well defined map on the double coset space. Let $y \in [[x]]$ and let $M \in G_\infty K_f$ be such that $y = M\cdot x$. By the local to global principle for Hermitian forms there exists $N \in G_E$ such that $y = N\cdot x$. Now let $h = N^{-1}M \in H^x_{\mathbb{A}}$ then clearly $\mathfrak{i}(h) = [y]$. This proves surjectivity. If $h_1, h_2 \in H^x_{\mathbb{A}}$ with respective decompositions $h_i = N_i^{-1}M_i$ are such that $[N_1\cdot x] = [N_2\cdot x]$ then there exists $\gamma \in G_{\mathcal{O}}$ such that $N_1\cdot x = (\gamma N_2)\cdot x$. Note also that $M_i\cdot x = N_i\cdot x$ and therefore we get that $N_1^{-1}\gamma N_2 \in H^x_{\mathbb{Q}}$, that $M_2^{-1}\gamma^{-1}M_1 \in (H^x_{\mathbb{A}_f}\cap K_f)H^x_\infty$ and that

$$h_1 = (N_1^{-1}\gamma N_2)h_2(M_2^{-1}\gamma^{-1}M_1).$$

This proves injectivity of $\mathfrak{i}$. Note that $h \mapsto \phi(h\theta)$ is a function on the double coset space $H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}/(H^x_{\mathbb{A}_f}\cap K_f)H^x_\infty$ and therefore that

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}} \phi(h\theta)dh = \mathrm{vol}((H^x_{\mathbb{A}_f}\cap K_f)H_\infty)\sum_t \frac{1}{\#(t^{-1}H^x_{\mathbb{Q}}t\cap(K_fH^x_\infty))}\phi(t\theta)$$

where the sum is over a set of representatives t in the double coset space $H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}/(H^x_{\mathbb{A}_f}\cap K_f)H^x_\infty$. Let $t = N^{-1}M$ be a decomposition as above, so that $\mathfrak{i}(t) = [N\cdot x]$. Then $\phi(t\theta) = \phi(M\theta) = \phi^+(M\cdot x) = \phi^+(N\cdot x) = \phi^+(\mathfrak{i}(t))$. Note also that

$$t^{-1}H^x_{\mathbb{Q}}t\cap(K_fH^x_\infty) = M^{-1}NH^x_{\mathbb{Q}}N^{-1}M\cap(K_fH^x_\infty)$$

is conjugate to

$$NH^x_{\mathbb{Q}}N^{-1}\cap M(K_fH^x_\infty)M^{-1}=H^{N\cdot x}_{\mathbb{Q}}\cap(K_fH^{N\cdot x}_\infty).$$

The latter equality is since $M_f\in K_f$ and $M_\infty\cdot x=N\cdot x$. But

$$H^{N\cdot x}_{\mathbb{Q}}\cap(K_fH^{N\cdot x}_\infty)=\mathcal{E}(N\cdot x)$$

and therefore

$$\#(t^{-1}H^x_{\mathbb{Q}}t\cap(K_fH^x_\infty))=\epsilon(N\cdot x)=\epsilon(\mathfrak{i}(t)).$$

The lemma now follows. □

3. Periods of Eisenstein series and representation numbers

3.1. **Eisenstein series classical and adelic.** Here we set up some notation and define the Eisenstein series that we consider. We will only consider Eisenstein series induced from characters on standard parabolic subgroups. Let $B=TU$ be the standard Borel subgroup of G with its standard Levi decomposition and let $P=MV$ be a parabolic of type $(n_1,\dots,n_t)$ containing B with its standard Levi decomposition. For integers $a\le b$ we denote $[a,b]=\{a,a+1,\dots,b\}$. Let

$$I_i=[n_1+\cdots+n_{i-1}+1,n_1+\cdots+n_i],\ i=1,\dots,t$$

be the segments determined by P and let

$$N_i=n_{i+1}+\cdots+n_t,\ i=1,\dots,t-1.$$

We will view $\mathbb{C}^t$ as a subspace of $\mathbb{C}^n$ as follows. For $\mu=(\mu_1,\dots,\mu_t)\in\mathbb{C}^t$, when convenient, we will also denote by μ the n-tuple $(\mu_1^{(n_1)},\dots,\mu_t^{(n_t)})$, where $a^{(m)}$ is the m-tuple $(a,\dots,a)$. From now on, we will always consider t-tuples μ so that $n_1\mu_1+\cdots+n_t\mu_t=0$. Denote $I_P^G(\mu)=\mathrm{Ind}_{P_{\mathbb{A}_E}}^{G_{\mathbb{A}_E}}(\mu)$ the representation of $G_{\mathbb{A}_E}$ parabolically induced from the character

$$\mathrm{diag}(m_1,\dots,m_t)\mapsto\prod_{i=1}^t|\det m_i|_{\mathbb{A}_E}^{\mu_i}$$

on $M_{\mathbb{A}_E}$. For $\varphi\in I_P^G(\mu)$ we consider the Eisenstein series

$$E_P(g,\varphi,\mu)=\sum_{\gamma\in P_E\backslash G_E}\varphi(\gamma g).$$

Let

$$\varphi_\mu(mvk)=\prod_{i=1}^t|\det m_i|^{\mu_i+\frac{1}{2}(n_{i+1}+\cdots+n_t-(n_1+\cdots+n_{i-1}))}$$

where $m = \operatorname{diag}(m_1, \ldots, m_t) \in M_{\mathbb{A}_E}$, $v \in V_{\mathbb{A}_E}$ and $k \in K$, be the K-invariant element of $I_P^G(\mu)$ normalized so that $\varphi_\mu(e) = 1$. Denote $E_P(g;\mu) = E_P(g, \varphi_\mu, \mu)$. Since the field E has class number one, the embedding of $G_{\mathcal{O}}$ in G_E defines a bijection $P_{\mathcal{O}}\backslash G_{\mathcal{O}} \simeq P_E\backslash G_E$. As a function on $G_{\mathcal{O}}\backslash X_\infty^+$, i.e. with $E^+(g \cdot e;\mu) = E_P(g;\mu)$, it can therefore be expressed by

(3.1)
$$E_P^+(x;\mu) = \det x^{\mu_1 + \frac{n_2+\cdots+n_t}{2}} \sum_{\delta \in P_{\mathcal{O}}\backslash G_{\mathcal{O}}} \prod_{i=1}^{t-1} d_{N_i}(\delta \cdot x)^{-(\mu_i - \mu_{i+1} + \frac{n_i + n_{i+1}}{2})}$$

where $d_i(x)$ is the determinant of the lower right $i \times i$ block of x. In particular, we have

$$E_B^+(x;\lambda) = \det x^{\lambda_1 + \frac{n-1}{2}} \sum_{\delta \in B_{\mathcal{O}}\backslash G_{\mathcal{O}}} \prod_{i=1}^{n-1} d_i(\delta \cdot x)^{-(\lambda_{n-i} - \lambda_{n-i+1} + 1)}. \tag{3.2}$$

The Eisenstein series $E_P^+(x;\mu)$ can be expressed as a residue of the Eisenstein series $E_B^+(x;\lambda)$. Whenever well-defined, we define the residue operator Res_P from functions on $\mathbb{C}^n$ to functions on $\mathbb{C}^t$ by

$$(\operatorname{Res}_P f)(\mu) = \lim_{\lambda \to \mu + \Lambda(P)} f(\lambda) \prod_{\substack{j \in [1,n-1] \\ j \notin \{n - N_i : i = 1, \ldots, t-1\}}} (\lambda_j - \lambda_{j+1} - 1)$$

where

$$\Lambda(P) = (\Lambda_{n_1}, \ldots, \Lambda_{n_t}) \text{ and } \Lambda_n = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}) \in \mathbb{C}^n.$$

It is well known that

$$\operatorname{Res}_G E^+(x;\cdot)(0) \equiv c_n$$

is a constant and by computations of Langlands in [Lan71] we have

$$c_n = \frac{((\zeta_E^*)_{-1})^{n-1}}{\zeta_E^*(2)\zeta_E^*(3)\cdots\zeta_E^*(n)}.$$

We also set

$$c(P) = \prod_{i=1}^t c_{n_i}.$$

Using Langlands computation it can then be shown that

$$\operatorname{Res}_P E^+(x;\cdot)(\mu) = c(P) E_P^+(x;\mu). \tag{3.3}$$

3.2. **Eisenstein series and representation numbers.** For $x \in X_{\mathbb{Q}}$ we let Q_x denote the Hermitian form associated with the matrix x, i.e. $Q_x(\xi) = {}^t\bar{\xi} x \xi$ for $\xi \in \mathbb{C}^n$. We let $x \in X_{\mathbb{Q}}$ be such that $x_\infty \in X_\infty^+$ and Q_x is integral (i.e. $Q_x(\xi) \in \mathbb{Z}$ for all $\xi \in \mathcal{O}^n$). We will show that for such x, the Eisenstein series $E_P^+(x;\mu)$ is a Dirichlet series in the variables $(\mu_1 - \mu_2, \dots, \mu_{t-1} - \mu_t)$. We interpret the coefficients in terms of a type of representation number, which counts certain points on the (partial) flag variety $P_E \backslash G_E$. To define the representation numbers we will use the Plücker coordinates of the flag variety. To any $g \in G_E$ we associate the vectors $v_1(g), \dots, v_{n-1}(g)$ where $v_i(g) \in E^{\binom{n}{i}}$ is the vector of all $i \times i$ minors in the bottom i rows of g. For a vector $v \in E^m$ we denote by $[v]$ its $E^\times$-orbit in the projective space $\mathbb{P}_E^{m-1}$. The map

$$B_E g \mapsto ([v_{N_1}(g)], \dots, [v_{N_{t-1}}(g)])$$

is an embedding

$$P_E \backslash G_E \hookrightarrow \prod_{i=1}^{t-1} \mathbb{P}_E^{\binom{n}{N_i}-1}.$$

It will be more convenient for us to use the identification $P_{\mathcal{O}} \backslash G_{\mathcal{O}} \simeq P_E \backslash G_E$ and work with integral coordinates. The map

$$g \mapsto (v_{N_1}(g), \dots, v_{N_{t-1}}(g))$$

also defines an embedding

$$P_{\mathcal{O}} \backslash G_{\mathcal{O}} \hookrightarrow \prod_{i=1}^{t-1} \left(\mathcal{O}^{\binom{n}{N_i}} / \mathcal{O}^\times \right).$$

We denote

$$\mathcal{I}(P;\mathcal{O}) =$$
$$\{(v_1, \dots, v_{t-1}) \in \prod_{i=1}^{t-1} \mathcal{O}^{\binom{n}{N_i}} : \exists g \in G_{\mathcal{O}},\ v_{N_i}(g) = v_i,\ \forall i = 1, \dots, t-1\}.$$

Thus, a $t-1$ tuple is in $\mathcal{I}(P;\mathcal{O})$ if it satisfies the relations imposed by the variety $P_E \backslash G_E$.

To define the representation numbers we need some more notation. For any matrix $g \in M_{n \times k}(E)$ and integers $1 \le i \le n$, $1 \le j \le k$ we denote, as usual, the $(i,j)^{\text{th}}$ component of g by g_{ij}. We extend this notation as follows. Let

$$I_m(n) = \{(i_1, \dots, i_m) \in \mathbb{Z}^m : 1 \le i_1 < \dots < i_m \le n\}.$$

For $i = (i_1, \dots, i_r) \in I_r(n)$ and $j = (j_1, \dots, j_q) \in I_q(k)$ we denote by $g_{ij} \in M_{r \times q}(E)$ the matrix so that $(g_{ij})_{lm} = g_{i_l j_m}$ for $l = 1, \dots, r$ and

$m = 1, \dots, q$. Later on it will also be convenient, when $q \le n$ to denote $g^{(j)} = g_{[n+1-q,n],j}$.

Note that for $g \in G_E$ the linear operator $\wedge^k g : E^{\binom{n}{k}} \to E^{\binom{n}{k}}$ is represented by the matrix $(\det g_{ij})_{i,\, j \in I_k(n)}$ with respect to the basis $E_i = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $i \in I_k(n)$ of $E^{\binom{n}{k}}$, where e_i, $i = 1, \dots, n$ is the standard basis of E^n. From now on when we write $\wedge^k g$ we will mean the matrix $(\det g_{ij})_{i, j \in I_k(n)}$.

The representation numbers that we consider are defined for positive integers $k_1, \dots, k_{t-1}$ by

$$(3.4)\quad r_P(x; k_1, \dots, k_{t-1}) = \\ \#\{(v_1, \dots, v_{n-1}) \in \mathcal{I}(P; \mathcal{O}) : Q_{\wedge^{N_i} x}(v_i) = k_i,\ i = 1, \dots, t-1\}.$$

For every integer D define the Dirichlet series

$$Z_P^{(D)}(x; s_1, \dots, s_{t-1}) = w_E^{-(t-1)} \sum_{(k_1 k_2 \cdots k_{n-1}, D) = 1} \frac{r_P(x; k_1, \dots, k_{t-1})}{k_1^{s_1} k_2^{s_2} \cdots k_{t-1}^{s_{t-1}}}.$$

We also define the genus representation numbers

$$(3.5)\qquad r_P(\mathrm{gen}(x); k_1, \dots, k_{t-1}) = \sum_{y \in [[x]]/\sim} \epsilon^{-1}(y) r_P(y; k_1, \dots, k_{t-1})$$

and the associated Dirichlet series

$$Z_P^{(D)}(\mathrm{gen}(x); s_1, \dots, s_{t-1}) = w_E^{-(t-1)} \sum_{(k_1 k_2 \cdots k_{t-1}, D) = 1} \frac{r_P(\mathrm{gen}(x); k_1, \dots, k_{t-1})}{k_1^{s_1} k_2^{s_2} \cdots k_{t-1}^{s_{t-1}}}.$$

If $D = 1$ we will sometimes omit the superscript.

We now express special values of the Eisenstein series (3.2) in terms of the Dirichlet series $Z_P(x; s_1, \dots, s_{t-1})$. We need the following two Lemmas. The first is an elementary exercise in computation of a determinant, which we leave to the reader.

Lemma 3.1. *Let A and B be $k \times n$ matrices with $k \le n$. Then*

$$\det(A^t B) = \sum_{j \in I_k(n)} \det(A^{(j)} B^{(j)}).$$

Lemma 3.2. *For $\delta \in G_{\mathcal{O}}$ we have*

$$d_i(\delta \cdot x) = Q_{\wedge^i x}(v_i(\delta)).$$

Proof. We parameterize the coordinates of the vector $v_i(\delta)$ by $(v_j)_{j \in I_i(n)}$ where $v_j = \det(\delta^{(j)})$. Note that

$$d_i(\delta \cdot x) = d_i(\delta x^t \bar{\delta}) = \det((\delta x)_{[n+1-i,n],[1,n]}{}^t(\bar{\delta}_{[n+1-i,n],[1,n]})).$$

By Lemma 3.1 we get that

$$d_i(\delta\cdot x)=\sum_{j\in I_i(n)}\det((\delta x)^{(j)}(\bar\delta)^{(j)}). \tag{3.6}$$

We apply Lemma 3.1 once more to obtain

$$\det((\delta x)^{(j)})=\sum_{k\in I_i(n)}\det(\delta^{(j)}x_{kj}). \tag{3.7}$$

Plugging (3.7) into (3.6) we obtain that

$$d_i(\delta\cdot x)=\sum_{j,k\in I_i(n)}v_k\bar v_j x_{kj}.$$

□

Applying Lemma 3.2 we may now rewrite (3.1) as

$$\begin{aligned}&\det x^{-(\mu_1+\frac{n_2+\cdots+n_t}{2})}E_P^+(x,\mu)\\ &= w_E^{-(t-1)}\sum_{(v_1,\dots,v_{t-1})\in\mathcal{I}(P;\mathcal{O})}\prod_{i=1}^{t-1}Q_{\wedge^{N_i}x}(v_i)^{-(\mu_i-\mu_{i+1}+\frac{n_i+n_{i+1}}{2})}\\ &= w_E^{-(t-1)}\sum_{k_1,\dots,k_{t-1}\geq 1}\frac{r_P(x;k_1,\dots,k_{t-1})}{k_1^{-(\mu_1-\mu_2+\frac{n_1+n_2}{2})}\cdots k_{t-1}^{-(\mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2})}}.\end{aligned}$$

We have proven

Proposition 3.1. *Let $x\in X_{\mathbb{Q}}$ be such that $x_\infty\in X_\infty^+$ and Q_x is integral. Then*

$$E_P^+(x;\mu)=$$
$$\det x^{\mu_1+\frac{n_2+\cdots+n_t}{2}}Z_P(x;\mu_1-\mu_2+\frac{n_1+n_2}{2},\dots,\mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2}).$$

3.3. **The unitary period of an Eisenstein series.** In [Off], we obtained the following formula for the unitary period of an Eisenstein series.

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}}E(h,\varphi,\lambda)=2^{-n}\operatorname{vol}((E_1\backslash(E_1)_{\mathbb{A}})^n)\sum_\nu J^x(\nu,\varphi,\lambda). \tag{3.8}$$

Here $E_1=\{a\in E : a\bar a=1\}$ and we view E_1^n as a subgroup of T. Denote also by T' the group of $n\times n$ diagonal matrices defined over $\mathbb{Q}$ and by $N=N_{E/\mathbb{Q}}$ the norm map from $E^\times$ to $\mathbb{Q}^\times$. The term $J^x(\nu,\varphi,\lambda)$ is a factorizable linear functional on $I_B^G(\lambda)$ parameterized by the group of Hecke characters ν on $T'_{\mathbb{A}}/N(T_{\mathbb{A}_E})$, i.e. characters ν of $T'_{\mathbb{Q}}\backslash T'_{\mathbb{A}}$ such that $\nu\circ N=\mathbf{1}_T$ is the trivial character on $T_{\mathbb{A}_E}$. Thus the sum on the right hand side of (3.8) is over the 2^n characters $\nu=(\nu_1,\dots,\nu_n)$ where

$\nu_i \in \{\mathbf{1}_{T'}, \eta\}$ where η is the quadratic Hecke character associated to $E/\mathbb{Q}$ by class field theory. Let θ be as in (2.1). Applying the linear functional to the right shift $R(\theta)\varphi_\lambda$ of φ_λ by θ we have

$$J^x(\nu, R(\theta)\varphi_\lambda, \lambda) = J^{x_p}(\nu_\infty, R(\theta)\varphi_\infty, \lambda) \prod_{p<\infty} J^{x_p}(\nu_p, \varphi_p, \lambda).$$

For a precise definition of $J^x(\nu, \varphi, \lambda)$ and its local factors, we refer to [Off]. Recall that all unitary groups H^x are inner forms. We fix once and for all a Haar measure on $H^e_{\mathbb{A}}$ and choose compatible measures on the other unitary groups. The volume element appears in the formula for the period because the J^x functionals on the right hand side are proportional to the volume of $H^x_{\mathbb{A}}$ and inverse proportional to the volume on $(E_1)^n_{\mathbb{A}}$. Globally, the functionals also satisfy

$$J^x(\eta\nu, \varphi, \lambda) = \eta(\det x) J^x(\nu, \varphi, \lambda) \tag{3.9}$$

where $\eta\nu = (\eta\nu_1, \dots, \eta\nu_n)$. We remark that up to a finite product of local terms, the right hand side of (3.8) is expressed explicitly as a Dirichlet series in the variables $(\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n)$. We recall here the explicit formulas that we know for the local terms. Let

$$J^{x_p}(\nu_p; \lambda) = \frac{\mathrm{vol}((E_1)^n_p \cap K_p)}{\mathrm{vol}(H^e_p \cap K_p)} J^{x_p}(\nu_p, \varphi_p, \lambda)$$

where φ_p is the K_p-invariant section in $\mathrm{Ind}_{B_p}^{G_p}(\lambda)$ normalized so that $\varphi_p(e) = 1$. If $p < \infty$ is either split or such that $E_p/\mathbb{Q}_p$ is an unramified quadratic extension of local fields then

$$J^{x_p}(\nu_p; \lambda) = P_{m(x_p)}(\lambda) \prod_{1 \le i < j \le n} \frac{L_p(\nu_i \nu_j \eta, \lambda_i - \lambda_j)}{L_p(\nu_i \nu_j, \lambda_i - \lambda_j + 1)} \tag{3.10}$$

where $P_m(\lambda)$ is a polynomial in $p^{\lambda_1}, \dots, p^{\lambda_n}$ that we can write explicitly. If $x_p \in K_p \cdot e$ then $P_{x_p}(\lambda) = 1$. For any $x_p \in X_p$ there exists a unique $m = m(x) = (m_1, \dots, m_n) \in \mathbb{Z}^n$ with $m_1 \ge \cdots \ge m_n$ such that $x_p \in K_p \cdot p^m$ where $p^m = \mathrm{diag}(p^{m_1}, \dots, p^{m_n})$. We then have

$$P_m(\lambda) = \nu_0(p^m) \frac{\prod_{i=1}^n L_p(\eta^i, i)}{L_p(\eta, 1)^n} \sum_{\sigma \in W} \sigma \left(p^{\langle \lambda - \Lambda_n, m \rangle} \prod_{i<j} \frac{L_p(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j)}{L_p(\nu_i \nu_j^{-1} \eta, \lambda_i - \lambda_j + 1)} \right) \tag{3.11}$$

where $\nu_0 = (\eta, \eta^2, \dots, \eta^n)$ and σ acts on λ by permuting the indices. Up to a constant depending on x_p, $P_{m(x_p)}(\lambda)$ is the m-th Hall-Littlewood polynomial evaluated at p^λ. In the case where $E_p/\mathbb{Q}_p$ is ramified there are no explicit formulas available for $J^{x_p}(\nu_p;\lambda)$, but if $x_p \in K_p \cdot e$ then we have an asymptotic formula

$$\lim_{\lambda\mapsto\infty} J^{x_p}(\nu_p;\lambda) = 2^{n-1}\,\mathrm{ch}_{\{\nu_0,\eta\nu_0\}}(\nu). \tag{3.12}$$

In any case $J^{x_p}(\nu_p,\varphi_p,\lambda)$ is a rational function in $p^{-\lambda_1},\dots,p^{-\lambda_n}$. The formulas (3.9), (3.10), (3.11) and (3.12) can be found in [Off]. In [LO] we also observed that

$$J^x_\infty(\nu, R(\theta)\varphi_{\lambda,\infty},\lambda) = \frac{\mathrm{vol}(H^e_\infty \cap K_\infty)}{\mathrm{vol}((E_1)^n_\infty \cap K_\infty)}.$$

We obtain that

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}} E(h\theta,\lambda) = 2^{-n}\frac{\mathrm{vol}((E_1\backslash(E_1)_{\mathbb{A}})^n)}{\mathrm{vol}((E_1)^n_{\mathbb{A}}\cap K)}\,\mathrm{vol}((H^x_{\mathbb{A}_f}\cap K_f)H^e_\infty)\times \tag{3.13}$$
$$\sum_\nu\left[\left(\prod_{p\nmid\Delta_E} P_{x_p}(\lambda)\right)\prod_{i<j}\frac{L^{S_E}(\nu_i\nu_j\eta,\lambda_i-\lambda_j)}{L^{S_E}(\nu_i\nu_j,\lambda_i-\lambda_j+1)}\prod_{p\mid\Delta_E} J^{x_p}(\nu_p;\lambda)\right]$$

where Δ_E is the discriminant of E and S_E is the set of all prime numbers that divide Δ_E.

Lemma 3.3.

$$\frac{\mathrm{vol}((E_1\backslash(E_1)_{\mathbb{A}})^n)}{\mathrm{vol}((E_1)^n_{\mathbb{A}}\cap K)} = w_E^{-n}.$$

Proof. The quotient of volumes is of course independent of a choice of measure on $(E_1)_{\mathbb{A}}$. We fix the decomposable Haar measure on $(E_1)_{\mathbb{A}}$ as chosen in [LO] with respect to an additive character $\psi = \psi_0\circ\mathrm{Trace}_{\mathrm{E}/\mathbb{Q}}$ where ψ_0 is an additive character on $\mathbb{Q}\backslash\mathbb{A}$. The local measure on $(E_1)_p$ is determined by the exact sequence $1\to(E_1)_p\to E_p^\times\to\mathbb{Q}_p^\times$ and the Haar measure $d_{E_p^\times}x = L(1,\mathbf{1}_{E_p^\times})\frac{d^{\psi_p}x}{|x|_{E_p}}$ (resp. $d_{\mathbb{Q}_p^\times}x = L(1,\mathbf{1}_{\mathbb{Q}_p^\times})\frac{d^{(\psi_0)_p}x}{|x|_{\mathbb{Q}_p}}$) on $E_p^\times$ (resp. $\mathbb{Q}_p^\times$), where $d^{\psi_p}x$ (resp. $d^{(\psi_0)_p}$) is the self dual Haar measure on E_p (resp. $\mathbb{Q}_p$) with respect to ψ_p (resp. $(\psi_0)_p$). As explained in [LO], if we set

$$\mathfrak{d}_{E_p} = \mathfrak{d}^{\psi}_{E_p} = \begin{cases}\mathrm{vol}(\mathcal{O}_{E_p}) & E_p \text{ non-archimedean},\\ \frac12\,\mathrm{vol}(\{x+iy : 0\le x,y\le 1\}) & E_p \text{ complex}\end{cases}$$

where the volume is taken with respect to d^{ψ_p} then $\prod_p \mathfrak{d}_{E_p} = |\Delta_E|^{-\frac12}$ is independent of ψ. By Ono's formula for the Tamagawa number of a torus [Ono66] we have $\mathrm{vol}(E_1\backslash(E_1)_{\mathbb{A}}) = 2L^*(1,\eta)$. By Dirichlet's class number formula

$$L^*(1,\eta) = \frac{2h_E}{w_E\,|\Delta_E|^{\frac12}}$$

where h_E is the class number of E. Since we assume class number one, we see that $L^*(1,\eta) = 2w_E^{-1}\,|\Delta_E|^{-\frac{1}{2}}$ and therefore that

$$\mathrm{vol}((E_1\backslash(E_1)_{\mathbb{A}})^n) = (4w_E^{-1}\,|\Delta_E|^{-\frac{1}{2}})^n.$$

The volume on the denominator can be computed as the product over all primes of its local counterparts. We leave it to the reader to verify that

$$\mathrm{vol}((E_1)_p^n \cap K_p) = \begin{cases} \mathfrak{d}_{E_p}^n & p \text{ is either split or unramified} \\ (2\mathfrak{d}_{E_p})^n & p=\infty \text{ or } p \text{ is a ramified prime.} \end{cases}$$

In all nine cases of CM-fields of class number one we have $\sum_{p|\Delta_E} 1 = 1$. We therefore have

$$\mathrm{vol}((E_1)_{\mathbb{A}}^n \cap K) = (4\,|\Delta_E|^{-\frac{1}{2}})^n.$$

□

Applying Lemma 2.1 and Lemma 3.3 to (3.13) we get that

$$\sum_{y\in[[x]]/\sim} \epsilon(y)^{-1}E_B^+(y;\lambda) = (2w_E)^{-n}\times \tag{3.14}$$

$$\sum_\nu \left[\prod_{p\nmid\Delta_E} P_{x_p}(\lambda)\left(\prod_{i<j}\frac{L_p(\nu_i\nu_j\eta,\lambda_i-\lambda_j)}{L_p(\nu_i\nu_j,\lambda_i-\lambda_j+1)}\right)\prod_{p|\Delta_E} J^{x_p}(\nu_p;\lambda)\right].$$

Combined with Proposition 3.1, (3.14) gives

(3.15)

$$Z_B(\mathrm{gen}(x);\lambda_1-\lambda_2+1,\dots,\lambda_{n-1}-\lambda_n+1) = (2w_E)^{-n}\det x^{-(\lambda_1+\frac{n-1}{2})}\times$$

$$\sum_\nu \left[\prod_{p\nmid\Delta_E} P_{x_p}(\lambda)\left(\prod_{i<j}\frac{L_p(\nu_i\nu_j\eta,\lambda_i-\lambda_j)}{L_p(\nu_i\nu_j,\lambda_i-\lambda_j+1)}\right)\prod_{p|\Delta_E} J^{x_p}(\nu_p;\lambda)\right].$$

Similarly, applying Res_P to (3.14) and taking (3.3) into consideration we have proven

Theorem 3.1. *Let $x \in X_{\mathbb{Q}}$ be such that $x_\infty \in X_\infty^+$ and Q_x is integral. Then for any parabolic subgroup P of G containing B we have*

$$Z_P(\mathrm{gen}(x);\mu_1-\mu_2+\frac{n_1+n_2}{2},\dots,\mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2}) =$$

$$(2w_E)^{-n}c(P)^{-1}\det x^{-(\mu_1+\frac{n_2+\cdots+n_t}{2})}\times$$

$$\mathrm{Res}_P\sum_\nu \left[\prod_{p\nmid\Delta_E} P_{x_p}(\lambda)\left(\prod_{i<j}\frac{L_p(\nu_i\nu_j\eta,\lambda_i-\lambda_j)}{L_p(\nu_i\nu_j,\lambda_i-\lambda_j+1)}\right)\prod_{p|\Delta_E} J^{x_p}(\nu_p;\lambda)\right].$$

If x is such that x_l is in the K_l-orbit of the identity for some prime $l \mid \Delta_E$ then we can obtain more explicit formulas for the representation numbers $r_P(x; k_1, \ldots, k_t)$, for integers k_i not divisible by l, by using the asymptotic formula (3.12). In view of (3.9) we have

Corollary 3.1. *If in addition to the assumptions in Theorem 3.1 we have $x_l \in K_l \cdot e$ where l is the unique prime dividing Δ_E then*

$$Z_P^{(\Delta_E)}(\mathrm{gen}(x); \mu_1 - \mu_2 + \frac{n_1 + n_2}{2}, \ldots, \mu_{t-1} - \mu_t + \frac{n_{t-1} + n_t}{2}) = w_E^{-n} c(P)^{-1} \det x^{-(\mu_1 + \frac{n_2 + \cdots + n_t}{2})} \prod_{p \nmid \Delta_E} P_{x_p}(\mu + \Lambda(P)) \times \mathrm{Res}_P \prod_{p \nmid \Delta_E} \left(\prod_{i<j} \frac{L_p(\eta^{i+j+1}, \lambda_i - \lambda_j)}{L_p(\eta^{i+j}, \lambda_i - \lambda_j + 1)} \right).$$

For $s = (s_1, \ldots, s_{t-1}) \in \mathbb{C}^{t-1}$ we set $\mu(s) = (\mu_1, \ldots, \mu_t) \in \mathbb{C}^t$ where

$$\mu_i = \frac{1}{n} \left[\sum_{j=1}^{t-1} N_j (s_j - \frac{n_j + n_{j+1}}{2}) \right] - \sum_{j=1}^{i-1} (s_j - \frac{n_j + n_{j+1}}{2}). \tag{3.16}$$

We then have $s_i = \mu_i - \mu_{i+1} + \frac{n_i + n_{i+1}}{2}$, $i = 1, \ldots, t-1$.

4. Explicit examples

4.1. **The mirabolic parabolic.** Assume here that P is the parabolic subgroup of G of type $(n-1, 1)$. As explained in §1, the representation number $r_P(x; k)$ is then the number of ways to represent k by the Hermitian form Q_x with primitive vectors. We also denote

$$r(x; k) = \#\{v \in \mathcal{O}^n : Q_x(v) = k\}$$

and

$$r(\mathrm{gen}(x); k) = \sum_{y \in [[x]]/\sim} \epsilon(y)^{-1} r(y; k).$$

Let

$$\hat{Z}^{(D)}(x; s) = w_E^{-1} \sum_{(k,D)=1} \frac{r(x; k)}{k^s}$$

and

$$\hat{Z}^{(D)}(\mathrm{gen}(x); s) = w_E^{-1} \sum_{(k,D)=1} \frac{r(\mathrm{gen}(x); k)}{k^s}.$$

Then it is easy to see that

$$\hat{Z}^{(D)}(x; s) = \zeta_E^{(D)}(s) Z_P^{(D)}(x; s)$$

and

$$\hat{Z}^{(D)}(\mathrm{gen}(x);s)=\zeta_E^{(D)}(s)Z_P^{(D)}(\mathrm{gen}(x);s).$$

Applying Corollary 3.1 and setting $\mu(s)=(\frac{s}{n}-\frac{1}{2},(1-n)(\frac{s}{n}-\frac{1}{2}))$ we get that whenever $x_l\in K_l\cdot e$ for $l\mid\Delta_E$ we have

$$\hat{Z}^{(\Delta_E)}(\mathrm{gen}(x);s)=w_E^{-n}\frac{\zeta_E^*(2)\zeta_E^*(3)\cdots\zeta_E^*(n-1)}{(\zeta_E^*)_{-1}^{n-2}}\det x^{-\frac{s}{n}} \tag{4.1}$$

$$\left(\frac{\zeta_{-1}^{(\Delta_E)}}{L^{(\Delta_E)}(\eta,2)}\right)^{n-2}\prod_{k=2}^{n-2}\left(\frac{L^{(\Delta_E)}(\eta^{k+1},k)}{L^{(\Delta_E)}(\eta^k,k+1)}\right)^{n-(k+1)}\prod_{p\nmid\Delta_E}P_{m(x_p)}(\mu(s)+\Lambda(P))$$

$$\zeta_E^{(\Delta_E)}(s)\prod_{i=1}^{n-1}\frac{L^{(\Delta_E)}(\eta^{i+n+1},s-i)}{L^{(\Delta_E)}(\eta^{i+n},s+1-i)}.$$

4.2. **The parabolic $(1,n-2,1)$.** Here we assume that $n\geq 3$ and that P is the standard parabolic subgroup of G of type $(1,n-2,1)$. The Plücker coordinates of a matrix g are given by

$$\begin{aligned} v=v_1(g) &= (v_1,v_2,\ldots,v_n)\\ w=v_{n-1}(g) &= (w_1,w_2,\ldots,w_n)\end{aligned}$$

where $v_i=g^{(i)}$ and $w_i=g^{([1,n]-\{i\})}$. We leave it to the reader to verify that

$$\mathcal{I}(P;\mathcal{O})=\{v,w\in\mathcal{O}_{prim}^n:\sum_{i=1}^n(-1)^iv_iw_i=0\}.$$

In order to interpret $r_P(x;m_1,m_2)$ as more familiar representation numbers we will use the change of variables $(v,w)\mapsto(v,w')$ where $w'=(w_1',\ldots,w_n')$ with $w_i'=(-1)^i\bar{w}_i$. Note then that

$$Q_{\wedge^{n-1}x}(w)=Q_{\det xx^{-1}}(w').$$

Therefore, the representation number $r_P(x;m_1,m_2)$ is the size of the set

$$\{v,w\in\mathcal{O}_{prim}^n:{}^t\bar{v}w=0,\,Q_{\det xx^{-1}}(w)=m_1,\,Q_x(v)=m_2\}.$$

Note further, that the map $(v,w)\mapsto(v,\det xx^{-1}w)$ is a bijection from this set to the set in (1.1). We also denote

$$r(x;m_1,m_2)=\{v,w\in\mathcal{O}^n:{}^t\bar{v}w=0,\,Q_{\det xx^{-1}}(w)=m_1,\,Q_x(v)=m_2\}$$

and $r(\mathrm{gen}(x);m_1,m_2)=\sum_{y\in[[x]]/\sim}\epsilon(y)^{-1}r(y;m_1,m_2)$. Let

$$\hat{Z}^{(D)}(x;s_1,s_2)=w_E^{-2}\sum_{(m_1m_2,D)=1}\frac{r(x;m_1,m_2)}{m_1^{s_1}m_2^{s_2}}$$

and

$$\hat{Z}^{(D)}(\mathrm{gen}(x); s_1, s_2) = w_E^{-2} \sum_{(m_1 m_2, D)=1} \frac{r(\mathrm{gen}(x); m_1, m_2)}{m_1^{s_1} m_2^{s_2}}$$

then it is easy to see that

$$\hat{Z}^{(D)}(x; s_1, s_2) = \zeta_E^{(D)}(s_1)\zeta_E^{(D)}(s_2) Z_P^{(D)}(x; s_1, s_2)$$

and

$$\hat{Z}^{(D)}(\mathrm{gen}(x); s_1, s_2) = \zeta_E^{(D)}(s_1)\zeta_E^{(D)}(s_2) Z_P^{(D)}(\mathrm{gen}(x); s_1, s_2).$$

Applying Corollary 3.1 and setting

$$\mu(s_1, s_2) = \left(\frac{(n-1)s_1 + s_2}{n} - \frac{n-1}{2}, \frac{s_2 - s_1}{n}, \frac{n-1}{2} - \frac{s_1 + (n-1)s_2}{n}\right)$$

we get that whenever $x_l \in K_l \cdot e$ for the prime $l \mid \Delta_E$ we have

(4.2)

$$\hat{Z}^{(\Delta_E)}(\mathrm{gen}(x); s_1, s_2) = w_E^{-n} \frac{\zeta_E^*(2)\zeta_E^*(3)\cdots\zeta_E^*(n-2)}{(\zeta_E^*)_{-1}^{n-3}} \det x^{-\frac{(n-1)s_1+s_2}{n}}$$

$$\left(\frac{\zeta_{-1}^{(\Delta_E)}}{L^{(\Delta_E)}(\eta, 2)}\right)^{n-3} \prod_{k=2}^{n-3} \left(\frac{L^{(\Delta_E)}(\eta^{k+1}, k)}{L^{(\Delta_E)}(\eta^k, k+1)}\right)^{n-(k+2)} \zeta_E^{(\Delta_E)}(s_1)\zeta_E^{(\Delta_E)}(s_2)$$

$$\prod_{p \nmid \Delta_E} P_{m(x_p)}(\mu(s_1, s_2) + \Lambda(P)) \frac{L^{(\Delta_E)}(\eta^n, s_1 + s_2 + 1 - n)}{L^{(\Delta_E)}(\eta^{n+1}, s_1 + s_2 + 2 - n)}$$

$$\prod_{i=2}^{n-1} \frac{L^{(\Delta_E)}(\eta^{i+n+1}, s_2 + 1 - i)}{L^{(\Delta_E)}(\eta^{i+n}, s_2 + 2 - i)} \frac{L^{(\Delta_E)}(\eta^i, s_1 + i - n)}{L^{(\Delta_E)}(\eta^{i+1}, s_1 + i + 1 - n)}.$$

Assume now that $n = 3$. We apply this formula to obtain an explicit expression for $r(e; m_1, m_2)$. We have

$$\sum_{(m_1 m_2, \Delta_E)=1} \frac{r(\mathrm{gen}(e); m_1, m_2)}{m_1^{s_1} m_2^{s_2}} =$$

$$w_E^{-1} \zeta^{(\Delta_E)}(s_1 - 1)\zeta^{(\Delta_E)}(s_1)\zeta^{(\Delta_E)}(s_2 - 1)\zeta^{(\Delta_E)}(s_2) \frac{L^{(\Delta_E)}(\eta, s_1 + s_2 - 2)}{\zeta^{(\Delta_E)}(s_1 s_1 + s_2 - 1)}.$$

We expand the right hand side as a Dirichlet series and equate coefficients with the Dirichlet series on the left hand side. Doing this, we find that whenever $\gcd(m_1 m_2, \Delta_E) = 1$,

$$r(\mathrm{gen}(e); m_1, m_2) = w_E^{-1} \sum_{d \mid gcd(m_1, m_2)} d\sigma_1(\frac{m_1}{d})\sigma_1(\frac{m_2}{d})\phi_\eta(d)$$

where

$$\phi_\eta(d) = \sum_{d_0|d} \mu(d/d_0)\eta(d_0)d_0 = d\prod_{p|d}\left(1 - \frac{\eta(p)}{p}\right)$$

is a twisted Euler function.

If the field E is such that $[[e]] = [e]$ (as is the case for example if $E = \mathbb{Q}(\sqrt{-1})$ or $E = \mathbb{Q}(\sqrt{-3})$ then we obtain explicitly the representation number $r(e; m_1, m_2)$. It is easy to see that $\mathcal{E}(e)$ consists of scaled permutation matrices with unit scales and therefore that $\epsilon(e) = 6w_E^3$. It follows that if E is a field of class number one for which the genus class of the identity consists of a unique class, then whenever m_1 and m_2 are relatively prime to the discriminant of E, the number $r(e; m_1, m_2)$ of pairs of orthogonal, $\mathcal{O}$-integral vectors lying on the complex 3-dimensional spheres of radius $\sqrt{m_1}$ and $\sqrt{m_2}$ respectively, is

$$6w_E^2 \sum_{d|gcd(m_1,m_2)} d\sigma_1(\frac{m_1}{d})\sigma_1(\frac{m_2}{d})\phi_\eta(d).$$

For $E = \mathbb{Q}(\sqrt{-1})$ and m_1m_2 odd, the number

$$96 \sum_{d|gcd(m_1,m_2)} d\sigma_1(m_1/d)\sigma_1(m_2/d)\phi_\eta(d)$$

counts the pairs of 6-tuples $(a_1, a_2, \ldots, a_6), (b_1, b_2, \ldots, b_6) \in \mathbb{Z}^6$ satisfying the equations

$$\begin{aligned} a_1^2 + a_2^2 + \cdots + a_6^2 &= m_1 \\ b_1^2 + b_2^2 + \cdots + b_6^2 &= m_2 \\ a_1b_1 + a_2b_2 + \cdots + a_6b_6 &= 0 \\ a_1b_2 - b_1a_2 + a_3b_4 - a_4b_3 + a_5b_6 - a_6b_5 &= 0. \end{aligned}$$

4.3. **The case of GL_4 and the Borel.** Assume here that $n = 4$. In this section we give an explicit description of the incidence relations and representation numbers arising from the minimal parabolic Eisenstein series. Our description of the incidence relations is taken from [BFH90].

Given a 4×4 matrix g and a subset S of $\{1, 2, 3, 4\}$ with r elements, we let $A_S(g) = \det g^{(S)}$ be the minor of the matrix obtained by taking the bottom r rows of g and the columns indexed by the elements of S.

Then the Plücker coordinates $v_i(g)$ are given by

$$\begin{aligned} v_1 &= {}^t(A_1, A_2, A_3, A_4) \\ v_2 &= {}^t(A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}) \\ v_3 &= {}^t(A_{123}, A_{124}, A_{134}, A_{234}). \end{aligned}$$

These coordinates satisfy the following incidence relations:

$$\begin{pmatrix} 0 & -A_{34} & A_{24} & -A_{23} \\ A_{34} & 0 & -A_{14} & A_{13} \\ -A_{24} & A_{14} & 0 & -A_{12} \\ A_{23} & -A_{13} & A_{12} & 0 \end{pmatrix} v_1 = 0 \tag{4.3}$$

$$\begin{pmatrix} 0 & -A_{12} & A_{13} & -A_{14} \\ A_{12} & 0 & -A_{23} & A_{24} \\ -A_{13} & A_{23} & 0 & -A_{34} \\ A_{14} & -A_{24} & A_{34} & 0 \end{pmatrix} v_3 = 0 \tag{4.4}$$

$$A_1A_{234} - A_2A_{134} + A_3A_{124} - A_4A_{123} = 0 \tag{4.5}$$

$$A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0 \tag{4.6}$$

Furthermore, for $g \in G_{\mathcal{O}}$ the vectors v_i are obviously *primitive:*

$$\begin{aligned} gcd(A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}) &= gcd(A_1, A_2, A_3, A_4) \\ &= gcd(A_{123}, A_{124}, A_{134}, A_{234}) = 1. \end{aligned} \tag{4.7}$$

Conversely, we have the following result.

Theorem 4.1 ([BFH90]). *If* $(v_1, v_2, v_3) \in \mathcal{O}^4 \times \mathcal{O}^6 \times \mathcal{O}^4$ *satisfies* (4.3),(4.4) *and* (4.7)*, then* $(v_1, v_2, v_3) \in \mathcal{I}(B, \mathcal{O})$*. In particular,* (4.5) *and* (4.6) *are automatically satisfied.*

This allows us to be explicit about the representation numbers arising from the $GL_4(\mathcal{O})$ minimal parabolic Eisenstein series. For $x \in X_{\mathbb{Q}}$ such that Q_x is integral, we have

$$\begin{aligned} r_B(x; j, k, l) = \#\{(v_1, v_2, v_3) \in \mathcal{O}^4 \times \mathcal{O}^6 \times \mathcal{O}^4 &: (4.3), (4.4)(4.7) \\ \text{are satisfied and } Q_x(v_1) = l, Q_{\wedge^2 x}(v_2) &= k, Q_{\wedge^3 x}(v_3) = j\}. \end{aligned} \tag{4.8}$$

When E is equal to the field of discriminant -4 or -3, the 4×4 identity matrix e is the only class in its genus [Iya69, Fei78]. Therefore in these cases we have

$$r_B(e; j, k, l) = 24 w_E^4 r_B(\text{gen}(e); j, k, l)$$

and using Corollary 3.1 and the relation (3.16) we get

$$(4.9)\quad Z_B^{(\Delta_E)}(e;s_1,s_2,s_3)= \\ 24\left[\frac{\zeta^{(\Delta_E)}(s_1-2)}{L^{(\Delta_E)}(\eta,s_1-1)}\frac{L^{(\Delta_E)}(\eta,s_1+s_2-1)}{\zeta^{(\Delta_E)}(s_1+s_2)}\frac{\zeta^{(\Delta_E)}(s_1+s_2+s_3)}{L^{(\Delta_E)}(\eta,s_1+s_2+s_3+1)}\right. \\ \left.\frac{\zeta^{(\Delta_E)}(s_2+1)}{L^{(\Delta_E)}(\eta,s_2+2)}\frac{L^{(\Delta_E)}(\eta,s_2+s_3+2)}{\zeta^{(\Delta_E)}(s_2+s_3+3)}\frac{\zeta^{(\Delta_E)}(s_3+1)}{L^{(\Delta_E)}(\eta,s_3+2)}\right].$$

Expanding out the Dirichlet series on the right hand side will give an expression for $r_B(e;j,k,l)$ when $\gcd(jkl,\Delta_E)=1$ in terms of divisor sums involving the Möbius function and the character η.

References

[BFH90] Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein. Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives. *Ann. of Math. (2)*, 131(1):53–127, 1990.

[Bor63] Armand Borel. Some finiteness properties of adele groups over number fields. *Inst. Hautes Études Sci. Publ. Math.*, (16):5–30, 1963.

[Bra41] Hel Braun. Zur Theorie der hermitischen Formen. *Abh. Math. Sem. Hansischen Univ.*, 14:61–150, 1941.

[EGM87] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke. Zeta-functions of binary Hermitian forms and special values of Eisenstein series on three-dimensional hyperbolic space. *Math. Ann.*, 277(4):655–708, 1987.

[Fei78] Walter Feit. Some lattices over $\mathbf{Q}(\sqrt{-3})$. *J. Algebra*, 52(1):248–263, 1978.

[Hir88a] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. I. *Japan. J. Math. (N.S.)*, 14(1):203–223, 1988.

[Hir88b] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. III. *Tohoku Math. J. (2)*, 40(4):651–671, 1988.

[Hir89] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. II. *Japan. J. Math. (N.S.)*, 15(1):15–51, 1989.

[Hir98] Yumiko Hironaka. Local zeta functions on Hermitian forms and its application to local densities. *J. Number Theory*, 71(1):40–64, 1998.

[Hir99] Yumiko Hironaka. Spherical functions and local densities on Hermitian forms. *J. Math. Soc. Japan*, 51(3):553–581, 1999.

[Hir00] Yumiko Hironaka. Classification by Iwahori subgroup and local densities on Hermitian forms. *Sūrikaisekikenkyūsho Kōkyūroku*, (1173):143–154, 2000. Automorphic forms, automorphic representations and automorphic L-functions over algebraic groups (Japanese) (Kyoto, 2000).

[Iya69] Kenichi Iyanaga. Class numbers of definite Hermitian forms. *J. Math. Soc. Japan*, 21:359–374, 1969.

[Lan71] Robert P. Langlands. *Euler products*. Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1.

[LO] Erez Lapid and Omer Offen. Compact unitary periods. Submitted.

[LR00] Erez Lapid and Jonathan Rogawski. Stabilization of periods of Eisenstein series and Bessel distributions on GL(3) relative to U(3). *Doc. Math.*, 5:317–350 (electronic), 2000.

[Off] Omer Offen. Stable relative Bessel distributions on $GL(n)$ over a quadratic extension. *Amer. J. Math.*. To appear.

[Ono66] Takashi Ono. On Tamagawa numbers. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 122–132. Amer. Math. Soc., Providence, R.I., 1966.

[Sch98] Alexander Schiemann. Classification of Hermitian forms with the neighbour method. *J. Symbolic Comput.*, 26(4):487–508, 1998.

[Sie35] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen. *Ann. of Math. (2)*, 36(3):527–606, 1935.

[Sie36] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen. II. *Ann. of Math. (2)*, 37(1):230–263, 1936.

[Sie37] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen. III. *Ann. of Math. (2)*, 38(1):212–291, 1937.

[SP04] Rainer Schulze-Pillot. Representation by integral quadratic forms—a survey. In *Algebraic and arithmetic theory of quadratic forms*, volume 344 of *Contemp. Math.*, pages 303–321. Amer. Math. Soc., Providence, RI, 2004.

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