Double Dirichlet Series and Theta Functions

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Dedicated to Professor Samuel J. Patterson
in honor of his sixtieth birthday

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Abstract

Generalized theta functions are residues of metaplectic Eisenstein series. Even in the case of the n-fold cover of $GL(2)$, the Fourier coefficients of these mysterious functions have not been determined beyond $n = 3$. However, a conjecture of Patterson illuminates the case $n = 4$. In this paper, we make a new conjecture concerning the Fourier coefficients of the theta function on the 6-fold cover of $GL(2)$, present some evidence for the conjecture, and prove it in the case that the base field is a rational function field. Though the conjecture involves a single complex variable, our approach makes critical use of double Dirichlet series.

1 Introduction

The quadratic theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$$

has been a familiar object since the 19^{th} century and it has found many applications in number theory and other fields. Weil observed that $\theta(z)$ can be interpreted as an automorphic form on the two-fold cover of $GL(2)$. An Eisenstein series $E^{(2)}(z, s)$ on this group can be constructed which has a pole at $s = 3/4$, and whose residue at this pole is a constant multiple of $\theta(z)$.

Kubota [Kub69] investigated automorphic forms on the corresponding n-fold cover of $GL(2)$, $n \geq 3$. He defined a metaplectic Eisenstein series $E^{(n)}(z,s)$ on this group whose constant coefficient has a pole at $s = 1/2 + 1/(2n)$. It follows that $E^{(n)}(z,s)$ has a pole at this point, and Kubota defined the n^{th}-order analog of the theta function, as

$$\theta^{(n)}(z) = \text{Res}_{s=1/2+1/(2n)} E^{(n)}(z,s).$$

The precise nature of this general n^{th} order theta function seems to be far more mysterious than the familiar $n = 2$ case. Patterson [Pat77a, Pat77b] determined (by means of a metaplectic converse theorem) that in the case $n = 3$ its Fourier coefficients are essentially cubic Gauss sums. Kazhdan and Patterson [KP84] then showed that on the n-cover of $GL(r)$ the Whittaker-Fourier coefficients of an analogously defined theta function satisfy certain periodicity properties. However, even for $GL(2)$, for $n \geq 4$ the Fourier coefficients of $\theta^{(n)}(z)$ have proved quite difficult to determine. Since they are naturally defined from an arithmetic situation (the n-fold cover is built using n^{th} power local Hilbert symbols; in some sense the existence of such a group is a reflection of n^{th} order reciprocity), it would be of great interest to do this, and such a determination would be likely to have applications. See, for example, [BBCFH06] for such an application which does not rely on a precise understanding of the coefficients.

Patterson [Pat84] (see also [EP92] for a correction and refinement of the original conjecture) has made a beautiful conjecture about the Fourier coefficients of $\theta^{(n)}(z)$ in the case $n = 4$. It was proved when the ground field is a rational function field in [Hof92], (see [Pat07] for a version of [Hof92] from Patterson's point of view). In addition, [We96], [EP92], extensive numerical investigations have been made in the cases $n = 4, 6$. But aside from some suggestions concerning the algebraic number field in which these coefficients ought to lie the values of the coefficients are not in general understood, even heuristically.

The purpose of this paper is to formulate a conjecture about some of the Fourier coefficients of $\theta^{(n)}(z)$ in the case $n = 6$, and to prove this conjecture in the case of a rational function field. This conjecture seems to be "almost" true in a more general setting than $n = 6$ but some extra insight is still missing.

In the next section we will set up some notation and explain in a rough, but hopefully informative way, what is known, what has been conjectured, and what is still inscrutable.

2 A formulation of the conjecture

Though a great deal of number theory is concerned with Euler products, constructions on the metaplectic group frequently give rise to Dirichlet series with analytic continuation and functional equation that are *not* Euler products. Remarkably, it is an observation of Patterson that the equality of two such series may nonetheless encode deep information about the Fourier coefficients of the higher order theta functions. In this section, we build on Patterson's insight to arrive at a new conjecture concerning $\theta^{(6)}$ that is formulated as such an equality, and we explain its consequences.

The series we require are Rankin-Selberg convolutions of metaplectic forms. Unfortunately, such convolutions require a great deal of care at bad places (indeed, even in the non-metaplectic case the treatment of such places is delicate). To avoid these difficulties we will work heuristically at first, following the style in the early sections of [Hof93]. We will then give a full, precise proof of the conjecture in the rational function field case in Section 6. One expects many aspects of the theory of automorphic forms over global fields to be uniform in terms of the base field, so the proof in this case is a likely indication of a more general phenomenon.

Let F be a global field containing the $2n^{\text{th}}$ roots of unity. Let $\mathfrak{o}$ denote the ring of integers of F. To give the heuristic treatment, we will imagine that the class number of $\mathfrak{o}$ is one and that all primes are unramified. These assumptions are never truly satisfied, but the S-integer formalism, introduced by Patterson in this context, allows one to make the heuristic definitions we give below precise. In addition to these simplifying assumptions, we will not keep track of powers of the numbers 2 and π in gamma factors, and we will neglect values of characters whose conductors consist of ramified primes (simplifying, for example, the statement of the Davenport-Hasse relation). A rational function field $\mathbb{F}_q(t)$ with q congruent to 1 modulo $4n$ comes close to satisfying these simplifying assumptions, and thus conjectures formulated via such simplifying assumptions can usually be stated, and occasionally proved, rigorously in this case. That is the situation with the conjecture we present below.

A fundamental object for us is the normalized Gauss sum with numerator m and denominator d formed with the j^{th} power of the k^{th} power residue symbol:

$$G_j^{(k)}(m,d) = \mathbf{N}d^{-1/2} \sum_{\alpha \pmod{d}} \left(\frac{\alpha}{d}\right)_k^j e\left(\frac{\alpha m}{d}\right),$$

where $e(x)$ is an additive character of conductor $\mathfrak{o}$ and $\mathbf{N}d$ denotes the absolute norm of d. With this normalization, $|G_j^{(k)}(m,d)| = 1$ when d is square-free and $(m,d) = 1$.

Because we will later work with both the n and $2n$-fold covers, let us begin with a discussion of the k-fold cover, $k \geq 2$. In this context, the m^{th} Fourier coefficient of Kubota's Eisenstein series consists of an arithmetic part times a Whittaker function (essentially a K-Bessel function with index $1/k$). The arithmetic part is a Dirichlet series

$$D_j^{(k)}(s,m) = \sum_d \frac{G_j^{(k)}(m,d)}{\mathbf{N}d^s}.$$

Here j is prime to k, and arbitrary; it may be regarded as parametrizing the different embeddings of the abstract group of k-th roots of unity into $\mathbb{C}^\times$. The sum is over d sufficiently congruent to 1. (More carefully, one would keep track of the dependence on the inducing data for the Eisenstein series and obtain a sum over non-zero ideal classes, see [BB06].) The product

$$\tilde{D}_j^{(k)}(s,m^2) = \Gamma_k(s)\zeta^*(ks - k/2 + 1)D_j^{(k)}(s,m^2) \tag{2.1}$$

has an analytic continuation and satisfies a functional equation

$$\mathbf{N}m^{s/2}\tilde{D}_j^{(k)}(s,m^2) = \tilde{D}_j^{(k)}(1-s,m^2)\mathbf{N}m^{(1-s)/2}. \tag{2.2}$$

Here

$$\Gamma_k(s) = \Gamma\left(s - \frac{1}{2} + \frac{1}{k}\right)\Gamma\left(s - \frac{1}{2} + \frac{2}{k}\right)\cdots\Gamma\left(s - \frac{1}{2} + \frac{k-1}{k}\right) \tag{2.3}$$

and ζ^* denotes the completed zeta function of the field F. The normalized series (2.1) is analytic except for simple poles at $s = 1/2 + 1/k, 1/2 - 1/k$, and its residue at $s = 1/2 + 1/k$ is given by

$$\mathrm{Res}_{s=1/2+1/k}\tilde{D}_j^{(k)}(s,m) = c\,\frac{\tau_j^{(k)}(m)}{\mathbf{N}m^{1/2k}}, \tag{2.4}$$

where c is a nonzero constant. The numerator $\tau_j^{(k)}(m)$ is the object we are investigating: the m^{th} Fourier coefficient of the theta function on the k-fold cover of $GL(2)$.

The Eisenstein series is an eigenfunction of the Hecke operators T_{p^k} for every prime p and consequently so is its residue, the theta function. This forces the $\tau_j^{(k)}(m)$ to obey certain Hecke relations (see [KP84, Hof93]). These are:

$$\tau_1^{(k)}(mp^i) = G_{i+1}^{(k)}(m,p)\tau_1^{(k)}(mp^{k-2-i}), \tag{2.5}$$

valid for $k \geq 2$, p a prime, $0 \leq i \leq k-2$, and $(m,p)=1$.

For the moment we will restrict ourselves to the case $m = 1$. Our object is to understand the nature of the coefficients $\tau_1^{(k)}(p^i)$, that is the coefficients at prime power indices of the theta function formed from the first power of the k^{th} order residue symbol. The periodicity relation proved by Kazhdan and Patterson reduces, in this case, to the relation

$$\tau_1^{(k)}(mp^k) = \mathbf{N}p^{1/2}\tau_1^{(k)}(m)$$

for any m. Thus when studying $\tau_1^{(k)}(p^i)$ we need go no higher than $i = n-1$.

Referring to (2.5) we see from taking $i = k-1$ that $\tau_1^{(k)}(p^{k-1}) = 0$. Also, from $i = k-2$ we see (normalizing so $\tau_1^{(k)}(1) = 1$), that

$$\tau_1^{(k)}(p^{k-2}) = G_{k-1}^{(k)}(1,p).$$

Thus the Hecke relations *completely determine* the coefficients in the cases $k = 2$, the familiar quadratic theta function, and $k = 3$, the cubic theta function whose coefficients were found by Patterson. In particular, when $k = 3$

$$\tau_1^{(3)}(p) = G_2^{(3)}(1,p) = \overline{G_1^{(3)}(1,p)}$$

and $\tau_1^{(3)}(p^2) = 0$.

Unfortunately, for $n \geq 4$, the information provided by the Hecke operators is incomplete. The first undetermined case, $n = 4$, was studied by Patterson in [Pat84]. The Hecke relations in this case give $\tau_1^{(4)}(p^3) = 0$ and

$$\tau_1^{(4)}(p^2) = G_3^{(4)}(1,p) = \overline{G_1^{(4)}(1,p)},$$

but $\tau_1^{(4)}(p)$ is just related to itself. When the m is reintroduced and we use periodicity we have the more refined information

$$\tau_1^{(4)}(mp) = G_2^{(4)}(m,p)\tau_1^{(4)}(mp).$$

The Gauss sum is

$$G_2^{(4)}(m,p) = \left(\frac{m}{p}\right)_4^2 G_2^{(4)}(1,p) = \left(\frac{m}{p}\right)_2 G_1^{(2)}(1,p) = \left(\frac{m}{p}\right)_2,$$

as the quadratic Gauss sum is trivial by our simplifying assumption. Thus $\tau_1^{(4)}(mp)$ must vanish unless m is a quadratic residue modulo p.

Patterson observed that there are two natural Dirichlet series that can be formed:

$$D_1(w) = \zeta(4w-1) \sum \frac{G_3^{(4)}(1,m)}{\mathbf{N}m^w}$$

and

$$D_2(w) = \zeta(4w-1) \sum \frac{\tau_1^{(4)}(m)^2}{\mathbf{N}m^w}.$$

The first is the first Fourier coefficient of the Eisenstein series on the 4-cover of $GL(2)$, multiplied by its normalizing zeta function, and with the variable change $2s-1/2 \to w$. The second is the Rankin-Selberg convolution of the theta function with itself (not its conjugate), also multiplied by its normalizing zeta factor. He conjectured that

$$D_2(w) = D_1(w)^2.$$

This conjecture was based on the fact that both sides had double poles in the same places, both had identical gamma factors, and when corresponding coefficients were matched, all provable properties of the coefficients of $D_2(w)$ were consistent with the completely known $D_1(w)$. If this conjecture were true it would follow that

$$\tau_1^{(4)}(m)^2 = G_3^{(4)}(1,m) \sum_{d_1 d_2 = m} \left(\frac{d_1}{d_2}\right)_2$$

and in particular that

$$\tau_1^{(4)}(p)^2 = 2G_3^{(4)}(1,p).$$

To date, Patterson's conjecture has remained unproved and even ungeneralized. A remarkable aspect of it is that it states that a naturally occurring Dirichlet series *without* an Euler product is equal to a square of another such Dirichlet series. In fact, one side (D_2) is the Rankin-Selberg convolution of a theta function on the 4-cover of $GL(2)$ with itself. The other side (D_1^2) is the square of a Rankin-Selberg convolution. The object being squared, D_1 is the first Fourier coefficient of the Eisenstein series on the 4-cover of $GL(2)$. Using [KP84], it may also be regarded as the analog of the standard L- series associated to the theta function on the 4-cover of $GL(3)$.

A weaker conjecture, that has been generalized, was made in [BH89]. It implies that $\tau_1^{(4)}(p)G_1^{(4)}(1,p) = \tau_3^{(4)}(p)$, i.e. that the argument of $\tau_1^{(4)}(p)$ is the square root of the conjugate Gauss sum. This was proved by Suzuki [Suz97] in the case where the ground field is a function field.

We now make a new conjecture relating Rankin-Selberg convolutions involving coefficients of the higher-order theta functions. We specify the 6-th order residue symbol by equation (3.6) below wth $n = 3$.

Conjecture 2.1

$$\zeta(3u-1/2)\sum \frac{\tau_1^{(6)}(m^2)}{\mathbf{N}m^u} = \zeta(3u-1/2)\sum \frac{G_1^{(3)}(1,d)}{\mathbf{N}d^u} \cdot \sum \overline{\frac{\tau_1^{(3)}(m)}{\mathbf{N}m^u}}.$$

The left hand side is the convolution of the theta function on the 6-fold cover of $GL(2)$ with the theta function on the double cover of $GL(2)$. The right hand side is the product of two terms: the first coefficient of the cubic Kubota Eisenstein series, multiplied by its normalizing zeta factor, and the Mellin transform of the theta function on the 3-fold cover of $GL(2)$. In this case $(n = 3)$ the two factors on the right are equal, but we write it this way with an eye toward potential future generalizations. We include the apparently extraneous zeta functions as they arise naturally in the normalizing factors.

Writing $m = m_1 m_2^2 m_3^3$, with m_1, m_2 square free and relatively prime, m_3 unrestricted we see by the periodicity properties of $\tau_1^{(6)}$ and the known valuation of $\tau_1^{(3)}$ that this conjectured equality translates to

$$\sum \frac{\tau_1^{(6)}(m_1^2 m_2^4)}{\mathbf{N}m_1^u \mathbf{N}m_2^{2u}} = \left(\sum \frac{G_1^{(3)}(1,d)}{\mathbf{N}d^u} \right)^2,$$

another striking identity involving the square of a series without an Euler product. Note that the Gauss sums $G_1^{(3)}(1,d)$ on the right hand side vanish unless d is square free.

If we cancel the zeta factor, and equate corresponding coefficients we have the following predicted behavior for the coefficients $\tau_1^{(6)}(m_1^2 m_2^4)$:

$$\tau_1^{(6)}(m_1^2 m_2^4) = G_1^{(3)}(1,m_2)^2 G_1^{(3)}(1,m_1) \left(\frac{m_2}{m_1}\right)_3^2 \sum_{m_1 = d_1 d_2} \left(\frac{d_2}{d_1}\right)_3.$$

Bearing in mind that $G_1^{(3)}(1,m_2)^2 = \overline{G_1^{(6)}(1,m_2)}$ by the Davenport-Hasse relation, this relation is consistent with what is implied by setting $k=6$ in the Hecke relations (2.5). Similarly all aspects of the identity given above are consistent with the Hecke relations. Setting $m_2 = 1$ and $m_1 = p$, this reduces to

$$\tau_1^{(6)}(p^2) = 2G_1^{(3)}(1,p).$$

The Conjecture is made after verifying that the polar behavior and gamma factors of the left and right hand sides are identical. This verification is the content of Sections 3, 4 and 5. Indeed, as will be seen, a similar conjecture is almost true for the general case where 3 is replaced by n and 6 by $2n$. The difficulty is that the identity is partially, but not completely, compatible with the Hecke relations.

3 A double Dirichlet series obtained from the $2n$-cover of $GL(2)$

We will obtain the desired information about the poles and gamma factors of the series above by first performing the easier task of determining the analytic continuation, polar lines and functional equations of several related multiple Dirichlet series. We begin by defining the following double Dirichlet series, initially for $\Re(s), \Re(w) > 1$. Let

$$Z_1(s,w) = \sum_{d,m} \frac{G_1^{(2n)}(m^2,d)}{\mathbf{N}d^s \mathbf{N}m^w}. \tag{3.1}$$

Also, for n odd, let

$$Z_2(s,w) = \sum_{d,m} \frac{G_1^{(n)}(m^2,d)}{\mathbf{N}d^s \mathbf{N}m^w}, \tag{3.2}$$

and for n even, let

$$Z_2(s,w) = \sum_{d,m} \frac{G_{n+1}^{(2n)}(m^2,d)}{\mathbf{N}d^s \mathbf{N}m^w}. \tag{3.3}$$

The corresponding normalized series are

$$\tilde{Z}_1(s,w) = \zeta^*(\delta n(s+w-1/2) - \delta n/2 + 1)\zeta^*(2ns - n + 1)Z_1(s,w) \tag{3.4}$$

and

$$\tilde{Z}_2(s,w) = \zeta^*(\delta ns - \delta n/2 + 1)\zeta^*(2ns + 2nw - 2n + 1)Z_2(s,w). \tag{3.5}$$

Here

$$\delta = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even,} \end{cases}$$

and the $*$ in the zeta functions again means that the appropriate gamma factors have been included.

Let χ_d and ψ_d be multiplicative characters of conductor d, with $\psi_d^2 = 1$. Then if $\tau(\chi)$ refers to the usual Gauss sum corresponding to χ, normalized to have absolute value 1, the Davenport-Hasse relation states (ignoring characters ramified at primes dividing $2n$) that

$$\tau(\chi_d)\,\tau(\chi_d\psi_d) = \tau(\chi_d^2).$$

We have also suppressed the quadratic Gauss sum as it is trivial with our simplifying assumptions. In the case n is odd we choose the $2n^{\text{th}}$ order power residue symbol

$$\left(\frac{\alpha}{d}\right)_{2n} = \left(\frac{\alpha}{d}\right)_n \left(\frac{\alpha}{d}\right)_2, \tag{3.6}$$

and so the Davenport-Hasse relation implies that

$$G_1^{(2n)}(1,d)\,G_1^{(n)}(1,d) = G_2^{(n)}(1,d). \tag{3.7}$$

In the case $n = 3$ this translates into the familiar

$$G_1^{(6)}(1,d) = G_2^{(3)}(1,d)\,\overline{G_1^{(3)}(1,d)} = \overline{G_1^{(3)}(1,d)^2}.$$

In the case n is even

$$\left(\frac{\alpha}{d}\right)_{2n}^{n+1} = \left(\frac{\alpha}{d}\right)_{2n} \left(\frac{\alpha}{d}\right)_2$$

and

$$\left(\frac{\alpha}{d}\right)_{2n}^{2} = \left(\frac{\alpha}{d}\right)_n,$$

so the Davenport-Hasse relation implies that

$$G_1^{(2n)}(1,d)\,G_{n+1}^{(2n)}(1,d) = G_1^{(n)}(1,d). \tag{3.8}$$

For example, if $n = 2$ this is the trivial relation

$$G_1^{(4)}(1,d)\,G_3^{(4)}(1,d) = G_1^{(2)}(1,d) = 1.$$

Our main tool in establishing the analytic continuation of the $Z_i(s,w)$, $i = 1, 2$, will be

Proposition 3.1 *For $\Re s, \Re w > 1$, both $Z_1(s,w)$ and $Z_2(s,w)$ converge absolutely. Furthermore, each has an analytic continuation for any fixed w as long as $\Re s$ is sufficiently large. In fact the following relations hold. For n odd*

$$\begin{aligned} Z_1(s,w) &= Z_2(s+w-1/2, 1-w) \\ &= \zeta(2ns+nw-n) \sum_{\tilde{d}_0} \frac{\overline{G(\chi_{\tilde{d}_0}^{(n)})} L(1-w, (\chi_{\tilde{d}_0}^{(n)})^2)}{(\mathbf{N}\tilde{d}_0)^{s+w/2} (\mathbf{N}d_0)^{(w-1)/2}} \\ &\times \prod_{(p,d_0)=1} \left(1 + \frac{\chi_{\tilde{d}_0}^{(n)}(p)}{\mathbf{N}p^{ns+(n-1)w/2-(n-1)/2}}\right) \\ &\times \prod_{(p,d_0)=1} \left(1 - \frac{\overline{\chi_{\tilde{d}_0}^{(n)}(p)}}{\mathbf{N}p^{ns+(n+1)w/2-(n-1)/2}}\right), \end{aligned}$$

and for n even

$$\begin{aligned} Z_1(s,w) &= Z_2(s+w-1/2, 1-w) \\ &= \zeta(2ns+nw-n) \sum_{\tilde{d}_0} \frac{\overline{G((\chi_{\tilde{d}_0}^{(2n)})^{n+1})} L(1-w, \chi_{\tilde{d}_0}^{(n)})}{(\mathbf{N}\tilde{d}_0)^{s+w/2} (\mathbf{N}d_0)^{(w-1)/2}} \\ &\times \prod_{(p,d_0)=1} \left(1 - \frac{1}{\mathbf{N}p^{2ns+nw-n+1}}\right). \end{aligned}$$

Here $G(\psi)$ refers to the Gauss sum associated to the character ψ, normalized to have absolute value equal to 1. The sums over d_0 and $\tilde{d}_0$ are defined as follows. If n is odd, then we write $\tilde{d}_0 = e_1 e_2^n$. Here e_1 is n^{th} power free, e_2 is the square free product of all p dividing e_1 such that the exact power of p dividing e_1 is even and we sum over all such e_1. If n is even then we sum over all $\tilde{d}_0$ that are $2n^{th}$ power free, with the proviso that if $p|\tilde{d}_0$ then an odd power of p must exactly divide $\tilde{d}_0$. We denote by d_0 the product of all the distinct primes dividing $\tilde{d}_0$.

Proposition 3.1 is proved by taking s, w to have large real parts, and interchanging the order of summation in $Z_1(s,w)$. A careful analysis reduces $Z_1(s,w)$ to the expressions on the right hand side above, but with the functional equation applied to the L-series in the numerator (i.e. the argument of the L-series is w rather than $1-w$.) The sum over d then converges absolutely for any fixed w as long as the real part of s is sufficiently large. If one applies the functional equation to the L-series and uses the the Davenport-Hasse relation the sum is transformed into that given in the Proposition. Similarly, if one takes $Z_2(s+w-1/2, 1-w)$, where $\Re(1-w)$ and $\Re s$ are sufficiently large to insure absolute convergence, and interchanges the order of summation, the right hand side of the Proposition is obtained directly.

One can alternatively take $Z_1(s,w), Z_2(s,w)$ and sum over d on the inside. If one does this, with the real parts of s, w sufficiently large, then one obtains

$$Z_1(s,w) = \sum_m \frac{D_1^{(2n)}(s, m^2)}{\mathbf{N}m^w} \tag{3.9}$$

and also

$$Z_2(s,w) = \sum_m \frac{D_{n-1}^{(n)}(s, m^2)}{\mathbf{N}m^w} \tag{3.10}$$

for n odd and

$$Z_2(s,w) = \sum_m \frac{D_{n-1}^{(2n)}(s, m^2)}{\mathbf{N}m^w} \tag{3.11}$$

for n even.

Applying the relations (3.9), (3.10), (3.11) and the functional equation (2.2) one obtains the following

Proposition 3.2 *For fixed s the series expressions* (3.9)*,* (3.10)*,* (3.11) *converge absolutely as long as the real part of w is sufficiently large. In the range of absolute convergence the normalized series $\tilde{Z}_1(s,w), \tilde{Z}_2(s,w)$ defined in* (3.4)*,*(3.5) *satisfy*

$$\tilde{Z}_1(s,w) = \tilde{Z}_1(1-s, w+2s-1)$$

and

$$\tilde{Z}_2(s,w) = \tilde{Z}_2(1-s, w+2s-1)$$

We are now in a position to obtain the analytic continuation of $\tilde{Z}_1(s,w)$ and $\tilde{Z}_2(s,w)$. First let us clear the poles by defining

$$\hat{Z}_i(s,w) = \mathcal{P}_i(s,w)\tilde{Z}_i(s,w) \tag{3.12}$$

for $i = 1, 2$, where

$$\mathcal{P}_1(s,w) = (s - \frac{1}{2} - \frac{1}{2n})(s - \frac{1}{2} + \frac{1}{2n})(w)(w-1)(w+2s-2)(w+2s-1) \\ \times (s+w-1-\frac{1}{\delta n})(s+w-1+\frac{1}{\delta n}) \tag{3.13}$$

and

$$\mathcal{P}_2(s,w) = \mathcal{P}_1(s+w-1/2, 1-w).$$

The factors in $\mathcal{P}$ are chosen to clear the poles in s and w in the region of absolute convergence, and also to satisfy $\mathcal{P}_i(s,w) = \mathcal{P}_i(1-s, w+2s-1)$ for $i = 1, 2$. Thus, in addition to being analytic in the region of absolute convergence,

$$\hat{Z}_i(s,w) = \hat{Z}_i(1-s, w+2s-1)$$

for $i = 1, 2$ and

$$\hat{Z}_1(s,w) = \hat{Z}_2(s+w-1/2, 1-w).$$

For $i = 1, 2$, $\hat{Z}_i(s,w)$ converges absolutely in the region $\Re s, \Re w > 1$. The functional equation in s given above in (2.2) implies a polynomial bound in $|m|^s$ for the Dirichlet series in the numerators of (3.9), (3.10), (3.11) when $\Re(s) < 0$. Consequently, the Phragmen-Lindelöf principle implies a bound for these series when $0 \le \Re(s) \le 1$. Thus $\hat{Z}_i(s,w)$ can be extended to a holomorphic function in the region in $\mathbb{C}^2$ given by

$$\{(s,w) \mid \Re(s) \le 0, \Re(w) > -2\Re(s) + 2\} \cup \{(s,w) \mid \Re(s) > 1, \Re(w) > 1\} \\ \cup \{(s,w) \mid 0 \le \Re(s) \le 1, \Re(w) > -\Re(s) + 2\}$$

Arguing similarly with the L-functions appearing in the representations of $Z_i(s,w)$ given in Proposition 3.1, the $\hat{Z}_i(s,w)$ extend holomorphically to the region

$$\{(s,w) \mid 0 \le \Re(w) \le 1, \Re(s) > -\Re(w)/2 + 3/2\} \\ \cup \{(s,w) \mid \Re(w) \le 0, \Re(s) > -\Re(w) + 3/2\}.$$

By Bochner's theorem, the functions $\hat{Z}_i(s,w)$ thus extend analytically to the convex closure of the union of these regions, which is the region

$$R_1 = \{(s,w) \mid s \le 0, \Re(w) > -2\Re(s) + 2\} \\ \cup \{(s,w) \mid 0 \le \Re(s) \le 3/2, \Re(w) > -4\Re(s)/3 + 2\} \\ \cup \{(s,w) \mid 3/2 \le \Re(s), \Re(w) > -\Re(s) + 3/2\}. \tag{3.14}$$

Applying the relation $\hat{Z}_1(s,w) = \hat{Z}_2(s+w-1/2, 1-w)$ we see that as the image of R_1 under the map $(s,w) \to (s+w-1/2, 1-w)$ intersects itself, we can extend both $\hat{Z}_1(s,w)$ and $\hat{Z}_2(s,w)$ to the convex hull of the union of R_1 and its image. This is the half plane

$$R_2 = \{(s,w) \in \mathbb{C}^2 \mid \Re(w) > -2\Re(s) + 2\}.$$

Finally, applying $\hat{Z}_i(s,w) = \hat{Z}_i(1-s, w+2s-1)$ for $i = 1,2$ and taking the convex hull of the union of overlapping regions we obtain analytic continuation to $\mathbb{C}^2$.

We summarize the above discussion in

Proposition 3.3 *The functions $\tilde{Z}_1(s,w)$ and $\tilde{Z}_2(s,w)$ defined in* (3.4), (3.5) *have an analytic continuation to all of $\mathbb{C}^2$, with the exception of certain polar lines. For $\tilde{Z}_1(s,w)$ these polar lines are $s = 1/2 \pm 1/(2n)$; $w = 1, 0$; $w + 2s - 1 = 1, 0$; $s + w - 1/2 = 1/2 \pm 1/(\delta n)$. For $\tilde{Z}_2(s,w)$ these polar lines are $s = 1/2 \pm 1/(\delta n)$; $w = 1, 0$; $w + 2s - 1 = 1, 0$; $s + w - 1/2 = 1/2 \pm 1/(2n)$.*

4 The residue of $\tilde{Z}_1(s,w)$ at $s = 1/2 + 1/(2n)$

Now that the analytic properties of $\tilde{Z}_1(s,w)$ have been established we can investigate the residue of this function at $s = 1/2 + 1/(2n)$. By (2.4) we have

$$\mathrm{Res}_{s=1/2+1/(2n)} Z_1(s,w) = \sum_m \frac{\tau_1^{(2n)}(m^2)}{\mathbf{N}m^{w+1/(2n)}}.$$

and

$$\mathrm{Res}_{s=1/2+1/(2n)} \tilde{Z}_1(s,w) = \zeta(\delta n w - \delta n/2 + \delta/2 + 1)\zeta(2) \sum_m \frac{\tau_1^{(2n)}(m^2)}{\mathbf{N}m^{w+1/(2n)}}.$$

Consequently we set $u = w + 1/(2n)$ and define

$$\tilde{L}(u) = \zeta(\delta n u - \delta n/2 + 1) \sum_m \frac{\tau_1^{(2n)}(m^2)}{\mathbf{N}m^{u}}. \tag{4.1}$$

Remark. This is one of the two Dirichlet series of interest to us. We have chosen to first derive the analytic properties of $\tilde{Z}_1(s,w)$ and then deduce the analytic properties of $\tilde{L}(u)$ by viewing this function as the residue of the two-variable Dirichlet series. It should be possible to analyze $\tilde{L}(u)$ directly by viewing it as a Rankin-Selberg convolution of the theta function on the $2n$-cover of $GL(2)$ with the quadratic theta function, but experience indicates that the two variable approach is considerably simpler to carry out.

By Proposition 3.3, $\tilde{L}(u)$ inherits an analytic continuation to $\mathbb{C}$ and a functional equation relating $\tilde{L}(u)$ to $\tilde{L}(1-u)$. Also, $\tilde{L}(u)$ is analytic except for possible poles at $u = 1 + 1/(2n), -1/(2n), 1 - 1/(2n), 1/(2n), 1/2 + 1/(\delta n), 1/2 - 1/(\delta n)$. Using the analytic properties of $\tilde{Z}_1(s, w)$ corresponding properties of $\tilde{L}(u)$ are derived as follows:

$$\begin{aligned}
&\lim_{u\to 1+1/(2n)} (u - 1 - 1/(2n))\tilde{L}(u) \\
&= \lim_{u\to 1+1/(2n)} (u - 1 - 1/(2n)) \lim_{s\to 1+1/(2n)} (s - 1 - 1/(2n))\tilde{Z}_1(s, u - 1/(2n)) \\
&\qquad\qquad = \lim_{s\to 1+1/(2n)} (s - 1 - 1/(2n)) \lim_{w\to 1} (w - 1)\tilde{Z}_1(s, w).
\end{aligned}$$

Thus we have approached the problem by interchanging the order of two limits. Using Proposition 3.1 above it is easy to compute that

$$\lim_{w\to 1} (w - 1)\tilde{Z}_1(s, w) = \zeta^*(\delta ns)\zeta^*(2ns - n + 1).$$

As $w = 1$ corresponds to $u = 1 + 1/(2n)$, we see that $\tilde{L}(u)$ will have a pole at $u = 1 + 1/(2n)$ (and at $u = -1/(2n)$) if and only if $\zeta^*(\delta ns)\zeta^*(2ns - n + 1)$ has a pole at $s = 1 + 1/(2n)$. As this is not the case, the potential pole of $\tilde{L}(u)$ at $u = 1 + 1/(2n)$ does not exist.

To investigate the behavior of $\tilde{L}(u)$ at u near $1 - 1/(2n)$ we consider $\lim_{w\to 2-2s} \hat{Z}_1(s, w)$. Applying the functional equations in sequence yields

$$\hat{Z}_1(s, w) = \hat{Z}_2(s+w-1/2, 1-w) = \hat{Z}_2(3/2-s-w, w+2s-1) = \hat{Z}_1(s, 2-2s-w)$$

from which we obtain

$$\begin{aligned}
\lim_{w\to 2-2s} \hat{Z}_1(s, w) &= -(s - \frac{1}{2} - \frac{1}{2n})(s - \frac{1}{2} + \frac{1}{2n})(2 - 2s)(1 - 2s) \\
&\quad \times (s - 1 - \frac{1}{\delta n})(s - 1 + \frac{1}{\delta n})\zeta^*(\delta n - \delta ns)\zeta^*(2ns - 1). \quad (4.2)
\end{aligned}$$

For behavior of $\tilde{L}(u)$ at u near $1/2 + 1/(\delta n)$, we likewise evaluate the limit $\lim_{w\to 1+1/(\delta n)-s} \hat{Z}_1(s, w)$. Applying the functional equations in sequence we obtain

$$\hat{Z}_1(s,w) = \hat{Z}_2(s+w-1/2, 2-2s-w).$$

Taking the limit as $w \to 1+1/(\delta n) - s$ yields

$$\begin{aligned}
\lim_{w\to 1+1/(\delta n)-s} \hat{Z}_1(s,w) &= (s-\frac{1}{2}-\frac{1}{2n})(s-\frac{1}{2}+\frac{1}{2n})(1+\frac{1}{\delta n}-s)(\frac{1}{\delta n}-s)\\
&\times (s-1+\frac{1}{\delta n})(s+\frac{1}{\delta n})(\frac{2}{\delta n})\zeta^*(2)\zeta^*(n+1-2ns)\\
&\times \lim_{s+w-\frac{1}{2}\to\frac{1}{2}+1/(\delta n)} (s+w-1-\frac{1}{\delta n})Z_2(s+w-\frac{1}{2}, 2-2s-w))\\
&= (s-\frac{1}{2}-\frac{1}{2n})(s-\frac{1}{2}+\frac{1}{2n})(1+\frac{1}{\delta n}-s)(\frac{1}{\delta n}-s)(s-1+\frac{1}{\delta n})\\
&\times (s+\frac{1}{\delta n})(\frac{2}{\delta n}))\zeta^*(2)\zeta^*(n+1-2ns)M^{(\delta n)}_{1+(\delta-1)n}(1-s). \quad (4.3)
\end{aligned}$$

Here

$$M^{(k)}_j(u) = \sum \frac{\tau^{(k)}_j(m)}{\mathbf{N}m^u} \quad (4.4)$$

denotes the Mellin transform of the theta function on the k-fold cover of $GL(2)$, with the underlying residue symbol being the j-th power of the standard one.

We have thus far computed $\hat{Z}_1(s, 2-2s)$ and $\hat{Z}_1(s, 1+1/(\delta n)-s)$. We will now evaluate these expressions as s approaches $1/2+1/(2n)$. Applying the relations (4.2) and (4.3) (and continuing to ignore primes dividing $2n$) we obtain for $n=2$:

$$\hat{Z}_1(\frac{3}{4}, \frac{1}{2}) = \kappa^2, \quad (4.5)$$

and for $n=3$:

$$\hat{Z}_1(\frac{2}{3}, \frac{2}{3}) = \kappa^2, \quad (4.6)$$

where $\kappa = \mathrm{Res}_{s=1}\zeta^*(s)$. For general $n \geq 4$ we obtain

$$\hat{Z}_1(\frac{1}{2}+\frac{1}{2n}, 1-\frac{1}{n}) = \zeta^*\left(\frac{\delta(n-1)}{2}\right)\zeta^*(n). \quad (4.7)$$

Translating back to $\tilde{L}(u)$, defined in (4.1) we see that as $u \to 1-1/(2n)$, for $n=2$

$$\tilde{L}(u) \sim \frac{\kappa^2}{(u-3/4)^2}, \quad (4.8)$$

for $n=3$

$$\tilde{L}(u) \sim \frac{\kappa^2}{(u-5/6)^2}, \tag{4.9}$$

and for general $n \geq 4$

$$\tilde{L}(u) \sim \frac{\zeta^*\left(\frac{\delta(n-1)}{2}\right)\zeta^*(n)}{(u-1+1/(2n))}. \tag{4.10}$$

In a similar manner we obtain, as $u \to 1/2 + 1/(\delta n)$, for $n \geq 4$

$$\tilde{L}(u) \sim \zeta^*(2) M^{(\delta n)}_{1+(\delta-1)n}\left(\frac{1}{2}-\frac{1}{2n}\right),$$

where $M^{(\delta n)}_{1+(\delta-1)n}(1-s)$ is defined in (4.4). Note that when $n=2, n=3$, the two poles coincide and create a double pole, while for all $n \geq 4$ these poles are separate. This may be related to the fact that the conjecture can be made consistent with the Hecke relations in only these two cases.

5 The gamma factors of $\tilde{L}(u)$ and a conjecture

Recall the gamma factors associated to $\tilde{D}^{(n)}_1(s, m^2)$ defined in (2.1) as $\Gamma_n(s)$:

$$\Gamma_n(s) = \Gamma\left(s-\frac{1}{2}+\frac{1}{n}\right)\Gamma\left(s-\frac{1}{2}+\frac{2}{n}\right)\cdots\Gamma\left(s-\frac{1}{2}+\frac{n-1}{n}\right).$$

Applying the functional equations of Proposition 2.2 in succession one sees that the gamma factors associated to $\tilde{Z}_1(s,w)$ are

$$\Gamma_{2n}(s)\Gamma_{\delta n}(s+w-1/2)\Gamma(w)\Gamma(w+2s-1).$$

Taking the residue at $s = 1/2 + 1/(2n)$ it follows that the gamma factors associated to $\tilde{L}(u)$ are

$$\Gamma_{\delta n}(u)\Gamma(u-\frac{1}{2n}))\Gamma(u+\frac{1}{2n}). \tag{5.1}$$

Recall that

$$M^{(k)}_j(u) = \sum \frac{\tau^{(k)}_j(m)}{\mathbf{N}m^u}$$

denotes the Mellin transform of the theta function on the k-fold cover of $GL(2)$, where the underlying residue symbol is raised to the j power. In contrast to the situation with $\tilde{L}(u)$ it is easy to verify directly that the gamma factors associated to $M_j^{(k)}(u)$ are $\Gamma(u-1/(2k))\Gamma(u+1/(2k))$. We therefore define

$$\tilde{M}_j^{(k)}(u)=\Gamma(u-\frac{1}{2k})\Gamma(u+\frac{1}{2k})M_j^{(k)}(u).$$

It is now apparent that the gamma factors associated to $\tilde{L}(u)$, given in (5.1), factor into those associated to $\tilde{D}_1^{(\delta n)}(u,1)$, namely $\Gamma_{\delta n}(u)$, times those associated to $\tilde{M}_j^{(n)}(u)$. (This is true for any j.)

Recall that for $n\geq 4$ the poles of $\tilde{L}(u)$ are simple and located at

$$u=1-1/(2n),1/(2n),1/2+1/(\delta n),1/2-1/(\delta n),$$

while in the cases $n=2,3$ they combine into double poles located at $u=3/4,5/6$. On the other hand $\tilde{D}_1^{(\delta n)}(u,1)$ has simple poles at $1/2+1/(\delta n),1/2-1/(\delta n)$, while it is easily verified that $\tilde{M}_j^{(n)}(u)$ has simple poles at $u=1-1/(2n),1/(2n)$.

Because of these observations, it is plausible to conjecture that for some value of j, $\tilde{L}(u)$ factors into a product $\tilde{M}_j^{(n)}(u)\tilde{D}_1^{(\delta n)}(u,1)$. We can investigate this more closely, by using the information provided by the Hecke operators, and conclude that a likely value for j is $j=1$. For example, after canceling gamma factors we might tentatively conjecture that the following Dirichlet series identities hold: for n odd

$$\zeta(nu-n/2+1)\sum\frac{\tau_1^{(2n)}(m^2)}{\mathbf{N}m^u}=\zeta(nu-n/2+1)\sum\frac{\overline{\tau_1^{(n)}(m)}}{\mathbf{N}m^u}\cdot\sum\frac{G_1^{(n)}(1,d)}{\mathbf{N}d^u}$$

and for n even

$$\zeta(2nu-n+1)\sum\frac{\tau_1^{(2n)}(m^2)}{\mathbf{N}m^u}=\zeta(2nu-n+1)\sum\frac{\overline{\tau_1^{(n)}(m)}}{\mathbf{N}m^u}\cdot\sum\frac{G_{n+1}^{(2n)}(1,d)}{\mathbf{N}d^u}.$$

Specializing to the case $n=2$ and canceling $\zeta(4u-1)$ this reduces to the relation

$$\sum\frac{\tau_1^{(4)}(m^2)}{\mathbf{N}m^u}=\sum\frac{\overline{\tau_1^{(2)}(m)}}{\mathbf{N}m^u}\cdot\sum\frac{G_3^{(4)}(1,d)}{\mathbf{N}d^u}.$$

Write $m=m_0m_1^2$, where m_0 is square free and m_1 is unrestricted. Then by the known properties of $\tau_1^{(4)}$ it follows that

$$\tau_1^{(4)}(m_0^2m_1^4)=\overline{G_1^{(4)}(1,m_0)}\mathbf{N}m_1^{1/2}$$

and thus the left hand side of the expression above equals

$$\sum \frac{\tau_1^{(4)}(m_0^2 m_1^4)}{\mathbf{N}m_0^u \mathbf{N}m_1^{2u}} = \zeta(2u-1/2) \sum \frac{G_3^{(4)}(1,d)}{\mathbf{N}d^u}.$$

As $\tau_1^{(2)}(m_0 m_1^2) = \mathbf{N}m_1^{1/2}$ if $m_0 = 1$ and vanishes otherwise, and as $G_3^{(4)}(1,d) = G_1^{(4)}(1,m_0)$ if $d = m_0$ is square free and vanishes otherwise, the identity holds in the case $n = 2$.

The case $n = 3$ has already been discussed in Section 2 after the formulation of Conjecture 2.1. When $n \geq 4$, the highest coefficient index before periodicity which comes into play is $\mathbf{N}p^{2n-2}$. At this index the Hecke relations confirm an equality of the left and right hand sides. Unfortunately they fail to confirm this equality at lower indices. The conjecture may thus need a mild modification to hold for $n \geq 4$, or it may fail completely. The question remains open.

6 A proof of the conjecture in the case of a rational function field and n odd

In this section we will work over the rational function field $\mathbb{F}_q(T)$. We will make crucial use of the paper [Hof92], in the sense that we will refer to it for all notation and a number of results. We require $q \equiv 1 \bmod n$, and for convenience, we also suppose that $q \equiv 1 \bmod 4$. The conjecture is provable in this case because over a function field any Dirichlet series with a functional equation (with finite conductor) must be a ratio of polynomials. The polar behavior of the Dirichlet series determines the denominator, and a finite amount of information about the early coefficients is enough to determine the numerator.

Let $n \geq 3$ be odd. The function field analog of the series $Z_1(s,w)$ above is the Rankin-Selberg convolution of $E^{(2n)}(z,u)$ with $\theta^{(2)}(z)$. In effect the theta function picks off the coefficients of the Eisenstein series with square index and assembles them in a Dirichlet series. The functional equation and polar behavior of the Dirichlet series are determined by the corresponding functional equations and polar behavior of the Eisenstein series in the integral:

$$\int E^{(2n)}(z,u)\, \theta^{(2)}(z)\, \overline{E^{(n)}(z,v)}\, d\mu(z),$$

where the integration is taken over a truncated fundamental domain. Although the integrand is not of rapid decay, the technique of *regularizing* the integral provides the functional equation and polar behavior of a Mellin transform of the part of the product $E^{(2n)}(z,u)$ with $\theta^{(2)}(z)$ that is of rapid decay. See [Z81] an exposition of this. The key point for us is that all the necessary information about the Mellin transform is determined by these properties.

Denoting this Mellin transform as $R(u,v)$, we have explicitly

$$R(u,v) = \int_{\mathrm{ord}(Y)\equiv 0 \bmod n} \sum c_m^{(2n)}(u,Y)\,\tau^{(2)}(m,Y)\,|Y|^{2v-2}\,d^\times Y, \tag{6.1}$$

where the sum is over $m \in A := \mathbb{F}_q[T]$ such that $-2-\deg m + 2\,\mathrm{ord}\,Y \geq 0$. Formulas for $c_m^{(2n)}$ are given in [Hof92]. In particular, if we let

$$D_m(u,i) = \sum_{\deg c \equiv i \bmod n} |c|^{-2u} g_1^{(2n)}(m,c),$$

then

$$c_m^{(2n)}(u,Y) = q|Y|^{2-2u}\tilde{D}_m(u,Y),$$

with

$$\tilde{D}_m(u,Y) = D_m(u,0)\Bigg(1 + (1-q^{-1}) \sum_{\substack{1\leq k \leq 2n\gamma-2-\deg m \\ k\equiv 0 \bmod n}} q^{k(1-2u)}\Bigg) \\ + D_m(u,1+\deg m)\, q^{2n\gamma-2-\deg m} g_{-1-\deg m}(\mu_m,T)\, q^{-2(2n\gamma-1-\deg m)u}. \tag{6.2}$$

Here μ_m denotes the leading coefficient of m. Note that $\tilde{D}_m(u,Y)$ is thus a non-zero constant plus a sum of positive powers of q^{-2u} that are multiples of n.

Let $\varpi = 1/T$ be the local uniformizer. The $\tau^{(2)}(m,Y)$ are the Fourier coefficients of the quadratic theta function, described by

$$\tau^{(2)}(m,Y) = \begin{cases} |Y|^{1/2} & m = m_0^2 \text{ with } \mathrm{ord}(\varpi^{-2}m_0^2Y^2)\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Substituting in to the integral (6.1), we can do the Y integration, obtaining

$$R(u,v) = c \sum_{n\gamma \geq 1+\deg m_0} q^{(2u-2v-1/2)n\gamma}\tilde{D}_{m_0^2}(u,\varpi^{n\gamma}),$$

where c is a nonzero constant.

Letting $s = 2u - 1/2, w = 2v - 2u + 1/2$, and denoting by $\tilde{R}(u, v)$, the product of $R(u, v)$ by the normalizing zeta and gamma factors of the two Eisenstein series, we have, corresponding to $\tilde{Z}_1(s, w)$,

$$\tilde{R}(u,v) = c_q q^{n-1-2ns}\zeta^*(2ns - n + 1)q^{n-1-ns-nw}\zeta^*(n(s + w - 1) + 1) \\ \times \sum_{n\gamma \geq 1+\deg m_0} q^{-wn\gamma}\tilde{D}_{m_0^2}(s/2 + 1/4, \varpi^{n\gamma}), \quad (6.3)$$

where c_q is a non-zero constant.

The functional equations of the Eisenstein series imply that $\tilde{Z}_1(s, w) = \tilde{R}(u, v)$ is a rational function of $x = q^{-s}$ and $y = q^{-w}$. Also there are, at most, simple poles at

$$s = 1/2 \pm 1/2n,\ w = 0, 1,\ w = 2 - 2s,\ w = 1 - 2s,\ s + w - 1/2 = 1/2 \pm 1/n.$$

We therefore write

$$\tilde{Z}_1(s, w) = \frac{P(x, y)}{D(x, y)}$$

with

$$D(x, y) = (1 - y^n)(1 - q^n y^n)(1 - q^{n-1}x^{2n})(1 - q^{n+1}x^{2n}) \\ (1 - q^{n+1}x^n y^n)(1 - q^{n-1}x^n y^n)(1 - q^n x^{2n} y^n)(1 - q^{2n}x^{2n}y^n). \quad (6.4)$$

Note from (6.3) that $\tilde{Z}_1(s, w)$ is of the form $x^{2n}(xy)^n y^n$ times a power series in x^n, y^n. Also, the functional equations of the Eisenstein series imply that

$$\tilde{Z}_1(s, w) = \tilde{Z}_1(s, 2 - 2s - w).$$

Combining this information with (6.4), we conclude that $P(x, y)$ is of the form

$$P(x, y) = x^{3n}y^{2n}\sum_{i=0}^{M}\sum_{j=0}^{N} B_{ij}x^{in}y^{jn}.$$

and satisfies the functional equation

$$P(x, y) = q^{6n}x^{6n}y^{6n}P(x, q^{-2}x^{-2}y^{-1}).$$

To go farther, we consider the residue

$$R(y; q) = \lim_{x^{2n}\to q^{-n-1}}(1 - q^{n+1}x^{2n})\tilde{Z}_1(s, w) = \frac{P(q^{-1/2-1/2n}, y)}{\mathcal{D}(y)},$$

where

$$\mathcal{D}(y) = (1-y^n)(1-q^ny^n)(1-q^{-2})(1-q^{(n+1)/2}y^n)(1-q^{(n-3)/2}y^n) \times (1-q^{-1}y^n)(1-q^{n-1}y^n). \quad (6.5)$$

Notice that $R(y;q)$ is a power series in y^n beginning with the power y^{2n}. Also, the functional equation above specializes to

$$P(q^{-1/2-1/2n}, y) = q^{9n} P(q^{-1/2-1/2n}, y^{-1}q^{-1+1/n}).$$

Let us introduce for clarity the (admittedly unnecessary) variable $t = yq^{-1/2n}$. For convenience, write $\tilde{R}(t;q) = R(y;q)$, $\tilde{P}(t) = P(q^{-1/2-1/2n}, y)$, and $\tilde{\mathcal{D}}(t) = \mathcal{D}(y)$, so

$$\tilde{R}(t;q) = \tilde{P}(t)\tilde{\mathcal{D}}(t).$$

The functional equation above in y becomes one sending $t \to q^{-1}t^{-1}$ and

$$\tilde{P}(t) = t^{6n}q^{3n}\tilde{P}(q^{-1}t^{-1}).$$

Thus if $\tilde{P}(t) = \sum_2^M B_i t^{ni}$, then the functional equation implies that $M = 4$ and $B_2 = q^{-n}B_4$. Also recall that B_2 is nonzero. Thus we arrive at the expression

$$\tilde{P}(t) = B_2 t^{2n}(1 + B_3' t^n + q^n t^{2n})$$

for certain coefficients B_2, B_3'.

Finally, we have that the residue of $\tilde{P}_1(t)$ is 0 at both $t^{-n} = q^{n+1/2}$ and at $t^{-n} = q^{-1/2}$. This forces

$$1 + B_3' t^n + q^n t^{2n} = (1 - q^{n+1/2}t^n)(1 - q^{-1/2}t^n).$$

Cancelling these two factors from the denominator $\tilde{\mathcal{D}}(t)$, we arrive at

Theorem 6.1 *The function $\tilde{R}(t;q)$ is of the form*

$$\tilde{R}(t;q) = \frac{c_{n,q}t^{2n}}{(1-q^{1/2}t^n)(1-q^{n/2+1}t^n)(1-q^{n/2-1}t^n)(1-q^{n-1/2}t^n)},$$

where $c_{n,q}$ is a nonzero constant.

Now we compare this to the Mellin transform computed in [Hof92]. The function $M_n(u;q)$ introduced in (5.2) there is defined as the Mellin transform of the theta function on the n-fold cover of $GL(2)$ over the function field $\mathbb{F}_q(T)$. The Mellin transform introduces a variable w. Continuing to let $y = q^{-w}$, we have

Proposition 6.2 *[Hof92] For a certain nonzero constant $c'_{n,q}$, one has*

$$M_n(y;q) = \frac{c'_{n,q} y^n}{(1-qy^n)(1-q^{2n-1}y^n)}.$$

Here M_n has functional equation

$$M_n(y;q) = M_n(y^{-1}q^{-2};q).$$

We also find the Dirichlet series part $D_n(t;q)$ of the Fourier coefficient of the n-th order metaplectic Eisenstein series in [Hof92], (5.2). From this equation, we have

$$D_n(t;q) = \frac{t^n}{(1-q^{n-1}t^n)(1-q^{n+1}t^n)},$$

with $t = q^{-2s}$. This function has functional equation under $s \mapsto 1-s$.

Let us compare these three expressions. We have

$$D_n(tq^{-1/2};q) = \frac{q^{-n/2}t^n}{(1-q^{n/2-1}t^n)(1-q^{n/2+1}t^n)}.$$

Also, suppose that q is an even power of the residue characteristic. Then we may compute the Mellin transform of the theta function over $\mathbb{F}_{q^{1/2}}(T)$. If we double the Mellin transform variable w to $2w$, then the resulting expression may still be expressed in terms of $y = q^{-w} = (q^{1/2})^{-2w}$. This is given by

$$M_n(y;q^{1/2}) = \frac{c'_{n,q^{1/2}} y^n}{(1-q^{1/2}y^n)(1-q^{n-1/2}y^n)}.$$

We thus find that, after normalizing so that the first coefficient of every power series equals 1,

Theorem 6.3 *Suppose that q is an even power of the residue characteristic. Then*

$$\tilde{R}(t;q) = M_n(t;q^{1/2})D_n(tq^{-1/2};q).$$

In other words, the rational polynomial on the left hand side that equals the Rankin Selberg convolution of the Eisenstein series on the $2n$-fold cover with the quadratic theta function, factors into the the rational polynomial representing the Mellin transform of a theta function on the n-fold cover times the first Fourier coefficient of the Eisenstein series on the n-fold cover.

This proves the conjecture in the case of the rational function field when n is odd. Unfortunately the conjecture is certainly not true over a number field for $n \geq 5$, as observed previously. Thus the special nature of the rational function field seems to give rise to too many simplifications! In particular, the numerators on both sides are (after cancellations) essentially trivial in this case. In a function field of higher genus, the numerators would be polynomials, and further structure would be revealed. It remains a very interesting open question to follow through the methods of this section in the case of *any* extension of the rational function field and to see what the actual relationship is between $\tilde{R}, M_n$ and D_n.

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