SUMS OF L-FUNCTIONS OVER RATIONAL FUNCTION FIELDS

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Abstract. Friedberg, Hoffstein and Lieman have constructed two related multiple Dirichlet series from quadratic and higher-order L-functions and Gauss sums. We compute these multiple Dirichlet series explicitly in the case of the rational function field. This is done by utilizing the functional equation of the L-functions and the functional equation relating the two multiple Dirichlet series. We also point out a very simple correspondence between these series and their p-parts.

1. Introduction

Let n be an integer greater than or equal to 2, $\mathbb{F}_q$ the finite field with q elements, and $K = \mathbb{F}_q(t)$ the rational function field. The main result of this paper is the explicit computation of an infinite sum of L-functions associated to n^{th} order Hecke characters of K. The infinite sums we consider are examples of double Dirichlet series in two complex variables, and can be written as power series in q^{-s} and q^{-w}. In fact it will turn out that the series we construct will be rational functions in q^{-s} and q^{-w}.

These series are function field analogs of the series studied by Friedberg, Hoffstein and Lieman in [FHL02]. In that paper, working over a number field F containing the n^{th} roots of unity, the authors study a double Dirichlet series that is roughly of the form

$$\sum_m L(s, \chi_m)(Nm)^{-w},$$

where the sum is over integral ideals m of F, the character χ_m is the n^{th} order power residue symbol associated to m, and Nm denotes the absolute norm. The authors show that this double Dirichlet series has a meromorphic continuation to all $(s, w) \in \mathbb{C}^2$ and satisfies a group of functional equations relating it to a second series constructed from Gauss sums. The main ingredients in the proof are the functional equation of $L(s, \chi_m)$, properties of the Fourier coefficients of the metaplectic Eisenstein series on the n-fold cover of GL_2, and Bochner's tube theorem.

In the case $n = 2$, these ideas were applied by Fisher and Friedberg [FF04] in the context of a general function field to show the rationality of double Dirichlet series constructed from quadratic L-functions. The case $n = 2$ is somewhat easier because the Gauss sum arising in the functional equation of a quadratic Hecke L-series is trivial, and the theory of metaplectic Eisenstein series is not needed.

Date: 26 May 2009.

The first named author is partially supported by a grant from the NSF.

In this paper, we follow a more elementary method originally introduced in [CFH06] in the case $n = 2$. We exploit the fact that

$$\sum_{\substack{d\in\mathbb{F}_q[t]\\ \deg d=k}} \left(\frac{d}{m}\right)$$

vanishes if k is bigger than or equal to the degree of m, unless m is a perfect n^{th} power. Here $\left(\frac{d}{m}\right) = \chi_m(d)$ denotes the n^{th} power residue symbol for m, d relatively prime. If m and d are monic, then we have the reciprocity law

$$\left(\frac{m}{d}\right) = \left(\frac{d}{m}\right), \tag{1.1}$$

when q is congruent to 1 mod $2n$, see e.g. Rosen [Ros02], Theorem 3.5.

We now describe our results more precisely. We will define two double Dirichlet series, explicitly compute them as rational functions in q^{-s}, q^{-w} and show that they satisfy functional equations that relate them to one another. We begin by defining two multiplicative weighting factors $a(d, m)$ and $b(d, m)$ for pairs of monic polynomials, as in [FHL02]. For a monic prime polynomial $\mathfrak{p}$, let

$$a(\mathfrak{p}^j, \mathfrak{p}^k) = \begin{cases} |\mathfrak{p}|^{(n-1)d/n} & \text{if } d = \min(j,k) \text{ and } d \equiv 0 \bmod n \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and

$$b(\mathfrak{p}^j, \mathfrak{p}^k) = \begin{cases} 1 & \text{if } k = 0 \\ |\mathfrak{p}|^{k/2-1}(|\mathfrak{p}| - 1) & \text{if } j \geq k, k \equiv 0 \bmod n, k > 0 \\ -|\mathfrak{p}|^{k/2-1} & \text{if } j = k-1, k \equiv 0 \bmod n, k > 0 \\ |\mathfrak{p}|^{(k-1)/2} & \text{if } j = k-1, k \not\equiv 0 \bmod n, k > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Then define

$$a(d, m) = \prod_{\substack{\mathfrak{p}^j||d\\ \mathfrak{p}^k||m}} a(\mathfrak{p}^j, \mathfrak{p}^k), \quad b(d, m) = \prod_{\substack{\mathfrak{p}^j||d\\ \mathfrak{p}^k||m}} b(\mathfrak{p}^j, \mathfrak{p}^k).$$

Here $|d|$ denotes the norm $q^{\deg d}$.

Let $\mathcal{O}$ denote $\mathbb{F}_q[t]$ and $\mathcal{O}_{mon}$ the set of monic polynomials in $\mathbb{F}_q[t]$. Let $\zeta_{\mathcal{O}}(s)$ be the zeta function of the ring $\mathcal{O}$, that is

$$\zeta_{\mathcal{O}}(s) = (1 - q^{1-s})^{-1}.$$

The first double Dirichlet series we consider is

$$Z_1(s, w) = \sum_{d,m\in\mathcal{O}_{mon}} \frac{\chi_{m_0}(\hat{d})a(d, m)}{|m|^w|d|^s}, \tag{1.4}$$

where m_0 is the n^{th} powerfree part of m and $\hat{d}$ is the part of d relatively prime to m_0. We show in Section 2 that this can be rewritten in terms of L-functions

$$Z_1(s, w) = \sum_{m\in\mathcal{O}_{mon}} \frac{L(s, \chi_{m_0})}{|m|^w} P(s; m), \tag{1.5}$$

where the $P(s; m)$ are finite Euler products defined in Proposition 2.1.

The second multiple Dirichlet series is built from Gauss sums. See Section 2 for the precise definition of the Gauss sum $g(r, \epsilon, \chi_m)$. Then

$$Z_2(s,w) = \zeta_{\mathcal{O}}(nw - \frac{n}{2} + 1) \sum_{d,m \in \mathcal{O}_{mon}} \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{\bar{\chi}_{m_0}(\hat{d}) b(d,m)}{|m|^w |d|^s} \tag{1.6}$$

where $m_\flat$ is the squarefree part of the n^{th} powerfree part of m.

We can now state our main theorems. The first describes a set of functional equations relating Z_1 and Z_2. Specifically, define

$$Z_1(s,w;\delta_i) = \sum_{\substack{d,m \in \mathcal{O}_{mon} \\ \deg m \equiv i \pmod n}} \frac{\chi_{m_0}(\hat{d}) a(d,m)}{|m|^w |d|^s}$$

and

$$Z_2(s,w;\delta_i) = \zeta_{\mathcal{O}}(nw - \frac{n}{2} + 1) \sum_{\substack{d,m \in \mathcal{O}_{mon} \\ \deg m \equiv i \pmod n}} \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{\bar{\chi}_{m_0}(\hat{d}) b(d,m)}{|m|^w |d|^s}.$$

Theorem 1.1. *We have the functional equation*

$$Z_1(s,w;\delta_i) = \begin{cases} q^{2s-1} \frac{1-q^{-s}}{1-q^{s-1}} Z_2(1-s, w+s-\frac{1}{2}; \delta_0) & \text{for } i = 0 \\ q^{2s-1} q^{1/2-s} \frac{\bar{\tau}(\epsilon^i)}{\sqrt{q}} Z_2(1-s, w+s-\frac{1}{2}; \delta_i) & \text{for } 0 < i < n. \end{cases}$$

The finite field Gauss sum $\tau(\epsilon^i)$ is defined in the following section.

This is proved in Section 4.

The second main theorem is the following:

Theorem 1.2. *The double Dirichlet series Z_1 and Z_2 are rational functions of $x = q^{-s}$ and $y = q^{-w}$. Explicitly,*

$$Z_1(s,w) = \frac{1 - q^2 xy}{(1-qx)(1-qy)(1-q^{n+1}x^n y^n)}, \tag{1.7}$$

and

$$Z_2(s,w) = \frac{1 - q^{3n/2} x^{n-1} y^n + \sum_{i=1}^{n-1} \left(\tau(\epsilon^i) q^{i-1+i/2} x^{i-1} y^i - \tau(\epsilon^i) q^{3i/2} x^i y^i \right)}{(1-qx)(1-q^{n/2+1}y^n)(1-q^{3n/2}x^n y^n)}. \tag{1.8}$$

This theorem is proved in Section 6.

We conclude this introduction with some remarks to put our results in the larger context of Weyl group multiple Dirichlet series. The general theory of Weyl group multiple Dirichlet series was introduced in [BBC+06] in order to unify and extend several constructions which had previously been studied. In particular the series Z_1 and Z_2 of the present paper are expected to be $n-2$ fold residues of the n variable Weyl group multiple Dirichlet series associated to the root system A_n. Brubaker and Bump [BB06] have verified this in the cubic case $n = 3$. This case is manageable because, thanks to Patterson [Pat77a, Pat77b], we have a complete understanding of the Fourier coefficients of the theta function on the 3-fold metaplectic cover of SL_2 which arise when we take the residues of the Dirichlet series constructed from cubic Gauss sums. For $n > 3$ the precise nature of the coefficients of the n-fold cover theta functions remains mysterious. Nevertheless, there is much evidence in favor of the expectation that the two series constructed by Friedberg, Hoffstein and Lieman

coincide with a multiresidue of a Weyl group multiple Dirichlet series. Indeed, one of the motivations of this paper is to lay the groundwork for investigating this question for $n > 3$ by explicitly computing and comparing the the relevant multiple Dirichlet series in the case of the rational functional field. For example, in [Chi08] the first named author has explicitly computed the cubic A_3 multiple Dirichlet series and checked that residues of this series give the two series in Theorem 1.2 of this paper when $n = 3$. See also [CG] for further examples of explicit computations of double Dirichlet series defined over the rational function field.

Finally we point out a curious connection between the series Z_1, Z_2 of Theorem 1.2 and their $\mathfrak{p}$-parts. Define the following generating series H_1, H_2 constructed from the respective $\mathfrak{p}$-parts of Z_1 and Z_2,

$$\begin{aligned} H_1(X,Y) &= \sum_{j,k\geq 0} a(\mathfrak{p}^j,\mathfrak{p}^k)X^jY^k, \text{ and} \\ H_2(X,Y) &= (1-|\mathfrak{p}|^{n/2-1}Y^n)^{-1}\sum_{j,k\geq 0} b(\mathfrak{p}^j,\mathfrak{p}^k)\frac{g(1,\epsilon,\chi_{\mathfrak{p}^k})}{\sqrt{|\mathfrak{p}_\flat^k|}}X^jY^k, \end{aligned} \tag{1.9}$$

where $X = |\mathfrak{p}|^{-s}$ and $Y = |\mathfrak{p}|^{-w}$. We will prove

$$\begin{aligned} H_1(X,Y) &= \frac{1-XY}{(1-X)(1-Y)(1-|\mathfrak{p}|^{n-1}X^nY^n)}, \\ H_2(X,Y) &= \frac{1-|\mathfrak{p}|^{n/2-1}X^{(n-1)}Y^n + \sum_{i=1}^{n-1}\frac{g(1,\epsilon^i,\chi_{\mathfrak{p}})}{\sqrt{|\mathfrak{p}|}}X^{(i-1)}Y^i|\mathfrak{p}|^{(i-1)/2}(1-X)}{(1-X)(1-|\mathfrak{p}|^{n/2-1}Y^n)(1-|\mathfrak{p}|^{n/2}X^nY^n)}. \end{aligned} \tag{1.10}$$

Note that the substitutions

$$\begin{aligned} X &\to qx, \\ Y &\to qy, \\ |\mathfrak{p}| &\to 1/q, \text{ and} \\ g(1,\epsilon^i,\mathfrak{p})/\sqrt{|\mathfrak{p}|} &\to \tau(\epsilon^i)/\sqrt{q} \quad \text{ for } 1 \leq i \leq r \end{aligned} \tag{1.11}$$

transform H_i into Z_i for $i = 1, 2$. This similarity between a rational function field multiple Dirichlet series and its $\mathfrak{p}$-part seems to hold in a much wider context, see e.g. [CFH06, Chi08].

2. Gauss sums and L-Functions

In this section we will define the Gauss sums and L-functions that are the constituents of our double Dirichlet series. We will mostly follow the notation of Patterson [Pat 2] but with some adjustments to facilitate comparison with [FHL02].

As in the introduction, K is the rational function field $\mathbb{F}_q(t)$ with polynomial ring $\mathcal{O} = \mathbb{F}_q[t]$. We let $\mathcal{O}_{mon}$ denote the subset of $\mathcal{O}$ consisting of monic polynomials and let $K_\infty = \mathbb{F}_q((t))$ denote the field of Laurent series in t^{-1}. Let $\mu_n = \{a \in \mathbb{F}_q : a^n = 1\}$ and let $\chi : \mathbb{F}_q^\times \to \mu_n$ be the character $a \mapsto a^{\frac{q-1}{n}}$.

In order to define Gauss sums we first need an additive character on K_∞. Let e_0 be a nontrivial additive character on the prime field $\mathbb{F}$ of $\mathbb{F}_q$. Use this to define a character $e_\star$ of $\mathbb{F}_q$ by $e_\star(a) = e_0(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}}a)$. Let ω be the global differential dx/x^2.

Finally define the character e of K_∞ by $e(y) = e_\star(\mathrm{Res}_\infty(\omega y))$ for $y \in K_\infty$. Note that

$$\{y \in K : e|y\mathcal{O} = 1\} = \mathcal{O}.$$

For any $c \in \mathcal{O}$, we'll use c_0 to indicate the n^{th}-power free part of c and $c_\flat$ for the squarefree part of c_0. Fix an embedding ϵ from the the n^{th} roots of unity of $\mathbb{F}_q$ to $\mathbb{C}^\times$. For $r, c \in \mathcal{O}$ we define the Gauss sum

$$g(r, \epsilon, \chi_c) = \sum_{y \mod c_\flat} \epsilon\left(\left(\frac{y}{c}\right)\right) e\left(\frac{ry}{c_\flat}\right).$$

We also need the Gauss sums associated to the finite field $\mathbb{F}_q$. These are defined by

$$\tau(\epsilon) = \sum_{j \in \mathbb{F}_q} \epsilon\left(j^{(q-1)/n}\right) e_0\left(j\right).$$

We define the L-function associated to χ_m by

$$L(s, \chi_m) = \sum_{d \in \mathcal{O}_{mon}} \chi_m(d)|d|^{-s}. \tag{2.1}$$

When m is n^{th}-power free, the L-function satisfies a functional equation that we will describe now. Denote the conductor of the character χ_m by $\operatorname{cond}\chi_m$. Thus

$$|\operatorname{cond}\chi_m| = \begin{cases} |m_\flat| & \deg m \equiv 0(n) \\ q|m_\flat| & \deg m \not\equiv 0(n). \end{cases}$$

Then the completed L-function

$$L^*(s, \chi_m) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_m) & \deg m \equiv 0(n) \\ L(s, \chi_m) & \deg m \not\equiv 0(n). \end{cases} \tag{2.2}$$

satisfies the functional equation

$$L^*(s, \chi_m) = q^{2s-1}|\operatorname{cond}\chi_m|^{1/2-s} \frac{g^*(1, \epsilon, \chi_m)}{|\operatorname{cond}\chi_m|^{1/2}} L^*(1-s, \bar{\chi}_m) \tag{2.3}$$

where

$$g^*(1, \epsilon, \chi_m) = \begin{cases} g(1, \epsilon, \chi_m) & \deg m \equiv 0 \pmod n \\ \bar{\tau}(\epsilon^i) g(1, \epsilon, \chi_m) & \deg m \equiv i \not\equiv 0 \pmod n. \end{cases}$$

From the functional equation, we see that $L(s, \chi_m)$ is a polynomial in q^{-s} whose degree is one less than the degree of $m_\flat$, if m is not a perfect n^{th} power. If $m = 1$, we recover the zeta function

$$\zeta_{\mathcal{O}}(s) = \sum_{d \in \mathcal{O}_{mon}} |d|^{-s} = \frac{1}{1-q^{1-s}}. \tag{2.4}$$

Expanding the components at infinity, we have the following functional equations when m is n^{th}-power free, $\deg m \equiv i \pmod n$:

$$L(s, \chi_m) = \begin{cases} q^{2s-1}|m_\flat|^{1/2-s} \frac{g(1,\epsilon,\chi_m)}{|m_\flat|^{1/2}} \frac{1-q^{-s}}{1-q^{-(1-s)}} L(1-s, \bar{\chi}_m) & i = 0 \\ q^{2s-1}(q|m_\flat|)^{1/2-s} \frac{\bar{\tau}(\epsilon^i)}{\sqrt{q}} \frac{g(1,\epsilon,\chi_m)}{|m_\flat|^{1/2}} L(1-s, \bar{\chi}_m) & 0 < i < n. \end{cases} \tag{2.5}$$

This functional equation will be used in Section 4 to relate Z_1 and Z_2.

We now introduce a modified L-function related to (2.1) by inserting the weighting factor $a(d,m)$. Define

$$L(s,\hat{\chi}_m)=\sum_{d\in\mathcal{O}_{mon}}\frac{\chi_{m_0}(\hat{d})a(d,m)}{|d|^s}, \tag{2.6}$$

where $\hat{d}$ is the part of d relatively prime to m_0. Since the weighting function is multiplicative, $L(s,\hat{\chi}_m)$ is an Euler product,

$$L(s,\hat{\chi}_m)=\prod_{\substack{\mathfrak{p}\in\mathcal{O}_{mon}\\ \text{irreducible}}}(1+\frac{\chi_{m_0}(\hat{\mathfrak{p}})a(\mathfrak{p},m)}{|\mathfrak{p}|^s}+\frac{\chi_{m_0}(\hat{\mathfrak{p}}^2)a(\mathfrak{p}^2,m)}{|\mathfrak{p}|^{2s}}+\ldots).$$

Further, since $a(d,m)=1$ when d and m are coprime, this Euler product agrees with the original L-function Euler product for all but finitely many places.

We will relate this modified L-function $L(s,\hat{\chi}_m)$ to $L(s,\chi_{m_0})$ and derive a bound on its degree as a polynomial in q^{-s}, as long as m is not a perfect n^{th}-power. These properties are given in the following proposition:

Proposition 2.1. *We have*

$$L(s,\hat{\chi}_m)=L(s,\chi_{m_0})P(s;m),$$

where $P(s;m)=\prod_{\mathfrak{p}}P_{\mathfrak{p}}(s;m)$ *and* $P_{\mathfrak{p}}(s;m)=$

$$\begin{cases}\left(1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}\right)\displaystyle\sum_{k=0}^{n\alpha-1}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}+|\mathfrak{p}|^{-n\alpha s}|\mathfrak{p}|^{(n-1)\alpha} & \text{if } \mathfrak{p}\nmid m_0\\ \displaystyle\sum_{k=0}^{n\alpha}\frac{a(\mathfrak{p}^{n\alpha+i},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} & \text{if } \mathfrak{p}^i||m_0 \text{ and } i\neq 0.\end{cases}$$

Here α *and* i *are the unique integers with* $0\leq i<n$ *and* $\mathfrak{p}^{n\alpha+i}\|m$. *In particular, for* m *not a perfect* n^{th} *power, the degree of* $L(s,\hat{\chi}_m)$ *as a polynomial in* q^{-s} *is less than the degree of* m.

Proof. Begin with the Euler product

$$\begin{aligned}L(s,\hat{\chi}_m)&=\prod_{\mathfrak{p}}\sum_{k=0}^{\infty}\frac{\chi_{m_0}(\hat{\mathfrak{p}}^k)a(m,\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}\\ &=\prod_{\mathfrak{p}^{n\alpha}||m}\sum_{k=0}^{\infty}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}\times\prod_{\substack{\mathfrak{p}^{n\alpha+i}||m\\ 0<i<n}}\sum_{k=0}^{\infty}\frac{a(\mathfrak{p}^{n\alpha+i},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}\end{aligned}$$

For primes $\mathfrak{p}$ with $i=0$—that is $\mathfrak{p}\nmid m_0$ and $\mathfrak{p}^{n\alpha}||m$, say— it follows from (1.2) that the tails of the sum are a geometric series with common ratio $\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}$. Thus for such $\mathfrak{p}$ the $\mathfrak{p}$-part is

$$\sum_{k=0}^{n\alpha-1}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}+\frac{|\mathfrak{p}|^{-n\alpha s}|\mathfrak{p}|^{(n-1)\alpha}}{1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}}=\left(1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}\right)^{-1}P_{\mathfrak{p}}(s;m),$$

where

$$P_{\mathfrak{p}}(s;m)=\sum_{k=0}^{n\alpha-1}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}\left(1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}\right)+|\mathfrak{p}|^{-n\alpha s}|\mathfrak{p}|^{(n-1)\alpha}. \tag{2.7}$$

For primes such that $\mathfrak{p}^i || m_0$ with $0 < i < n$, it follows from (1.2) that $a(\mathfrak{p}^{n\alpha+i}, \mathfrak{p}^k) = 0$ for $k > n\alpha$, so the $\mathfrak{p}$-part is a finite sum

$$P_{\mathfrak{p}}(s; m) = \sum_{k=0}^{n\alpha} \frac{a(\mathfrak{p}^{n\alpha+i}, \mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}.$$

Thus

$$L(s, \hat{\chi}_m) = L(s, \chi_{m_0}) P(s; m)$$

as claimed. The bound on the degree of $L(s, \hat{\chi}_m)$ follows from the bound on the degree of $L(s, \chi_{m_0})$ for $m_0 \neq 1$ and the degrees of the $P_p(s; m)$. □

3. Functional Equation: $H_1 \to H_2$

Recall that the generating series $H_1(X, Y)$ and $H_2(X, Y)$ of (1.9) define the p-parts of Z_1 and Z_2, respectively. We describe the functional equations relating $H_1(X, Y)$ and $H_2(X, Y)$. These will be used to prove the global functional equation relating Z_1 to Z_2.

The functional equations are a direct consequence of the following proposition:

Proposition 3.1. *The generating series $H_1(X, Y)$ and $H_2(X, Y)$ are rational functions of X and Y. Explicitly,*

$$H_1(X, Y) = \frac{1 - XY}{(1 - X)(1 - Y)(1 - |\mathfrak{p}|^{n-1} X^n Y^n)}, \tag{3.1}$$

and

$$H_2(X, Y) = \frac{1 - |\mathfrak{p}|^{n/2-1} X^{(n-1)} Y^n + \sum_{i=1}^{n-1} \frac{g(1, \epsilon^i, \chi_{\mathfrak{p}})}{\sqrt{|\mathfrak{p}|}} X^{(i-1)} Y^i |\mathfrak{p}|^{(i-1)/2} (1 - X)}{(1 - X)(1 - |\mathfrak{p}|^{n/2-1} Y^n)(1 - |\mathfrak{p}|^{n/2} X^n Y^n)}. \tag{3.2}$$

Proof. Equation (3.1) is obvious from the definition (1.2) of the $a(\mathfrak{p}^k, \mathfrak{p}^l)$. The evaluation of $H_2(X, Y)$ is simply a matter of recognizing geometric series. From the definitions of $b(\mathfrak{p}^j, \mathfrak{p}^k)$ in (1.3) and H_2 in (1.9), we have

$$\begin{aligned}
(1 - |\mathfrak{p}|^{n/2-1} Y^n) H_2(X, Y) = & \sum_{j=0}^{\infty} \frac{g(1, \epsilon, \chi_{\mathfrak{p}^0})}{\sqrt{|\mathfrak{p}_\flat^0|}} X^j Y^0 \\
& + \sum_{\alpha=1}^{\infty} \sum_{j=n\alpha}^{\infty} |\mathfrak{p}|^{n\alpha/2-1} (|\mathfrak{p}| - 1) \frac{g(1, \epsilon, \chi_{\mathfrak{p}^{n\alpha}})}{\sqrt{|\mathfrak{p}_\flat^{n\alpha}|}} X^j Y^{n\alpha} \\
& + \sum_{\alpha=1}^{\infty} -|\mathfrak{p}|^{n\alpha/2-1} \frac{g(1, \epsilon, \chi_{\mathfrak{p}^{n\alpha}})}{\sqrt{|\mathfrak{p}_\flat^{n\alpha}|}} X^{n\alpha-1} Y^{n\alpha} \\
& + \sum_{\alpha=0}^{\infty} \sum_{i=1}^{n-1} |\mathfrak{p}|^{(n\alpha+i-1)/2} \frac{g(1, \epsilon, \chi_{\mathfrak{p}^{n\alpha+i}})}{\sqrt{|\mathfrak{p}_\flat^{n\alpha+i}|}} X^{n\alpha+i-1} Y^{n\alpha+i}.
\end{aligned}$$

Evaluating the geometric series yields

(3.3)
$$(1-|\mathfrak{p}|^{n/2-1}Y^n)H_2(X,Y) = \frac{1}{1-X} + \frac{|\mathfrak{p}|^{n/2-1}(|\mathfrak{p}|-1)X^nY^n}{(1-X)(1-|\mathfrak{p}|^{n/2}X^nY^n)} + \frac{-|\mathfrak{p}|^{n/2-1}X^{n-1}Y^n}{1-|\mathfrak{p}|^{n/2}X^nY^n} + \sum_{i=1}^{n-1} \frac{\frac{g(1,\epsilon^i,\chi_{\mathfrak{p}})}{\sqrt{|\mathfrak{p}|}}|\mathfrak{p}|^{(i-1)/2}X^{i-1}Y^i}{(1-|\mathfrak{p}|^{n/2}X^nY^n)}.$$

Equation (3.2) follows by rewriting equation (3.3). □

For $0 \leq i < n$, define

(3.4)
$$H_1(X,Y;\delta_i) = \sum_{\substack{j,k\geq 0\\ k\equiv i(n)}} a(\mathfrak{p}^j,\mathfrak{p}^k)X^jY^k,$$
$$H_2(X,Y;\delta_i) = (1-|\mathfrak{p}|^{n/2-1}Y^n)^{-1} \sum_{\substack{j,k\geq 0\\ k\equiv i(n)}} b(\mathfrak{p}^j,\mathfrak{p}^k)\frac{g(1,\epsilon,\chi_{\mathfrak{p}^k})}{\sqrt{|\mathfrak{p}_\flat^k|}}\bar{\chi}_{\mathfrak{p}^k}(\hat{\mathfrak{p}}^j)X^jY^k.$$

We have shown in Proposition 3.1 that H_1 and H_2 are both rational functions in $|\mathfrak{p}|^{-s}$ and $|\mathfrak{p}|^{-w}$ and it is clear from this proposition that

$$H_1(X,Y;\delta_i) = \begin{cases} \frac{1-XY^n}{(1-X)(1-Y^n)(1-|\mathfrak{p}|^{n-1}X^nY^n)} & i\equiv 0 \\ \frac{(1-X)Y^i}{(1-X)(1-Y^n)(1-|\mathfrak{p}|^{n-1}X^nY^n)} & i\not\equiv 0, \end{cases} \tag{3.5}$$

and

$$H_2(X,Y;\delta_i) = \begin{cases} \frac{1-|\mathfrak{p}|^{n/2-1}X^{(n-1)}Y^n}{(1-X)(1-|\mathfrak{p}|^{n/2-1}Y^n)(1-|\mathfrak{p}|^{n/2}X^nY^n)} & i\equiv 0 \\ \frac{\frac{g(1,\epsilon^i,\chi_{\mathfrak{p}})}{\sqrt{|\mathfrak{p}|}}X^{(i-1)}Y^i|\mathfrak{p}|^{(i-1)/2}(1-X)}{(1-X)(1-|\mathfrak{p}|^{n/2-1}Y^n)(1-|\mathfrak{p}|^{n/2}X^nY^n)} & i\not\equiv 0. \end{cases} \tag{3.6}$$

The following theorem establishes a functional equation relating H_1 to H_2:

Theorem 3.2. *We have the functional equation*

$$H_1(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_i) = \begin{cases} \frac{1-|\mathfrak{p}|^{-(1-s)}}{1-|\mathfrak{p}|^{-s}}H_2(\mathfrak{p}^{-(1-s)},\mathfrak{p}^{-(w+s-1/2)};\delta_0) & i=0 \\ \frac{\sqrt{|\mathfrak{p}|}}{g(1,\epsilon^i,\chi_{\mathfrak{p}})}|\mathfrak{p}|^{s-1/2}H_2(\mathfrak{p}^{-(1-s)},\mathfrak{p}^{-(w+s-1/2)};\delta_i) & 0<i<n. \end{cases}$$

Proof. The proof is by a direct computation using Equations (3.5) and (3.6). □

4. Functional Equation: $Z_1 \to Z_2$

There is a set of functional equations relating Z_1 and Z_2. These will be described in this section. Define

$$Q(s;m) = \frac{P(1-s;m)}{(m/m_\flat)^{s-1/2}}. \tag{4.1}$$

With the expansion of P as an Euler product, we see that Q is also an Euler product supported in the primes dividing m:

$$Q(s;m) = \prod_{\substack{\mathfrak{p}^{n\alpha+i}||m\\ i=0}} \frac{1}{|\mathfrak{p}|^{n\alpha(s-1/2)}}P_{\mathfrak{p}}(1-s;m) \times \prod_{\substack{\mathfrak{p}^{n\alpha+i}||m\\ 0<i<n}} \frac{1}{|\mathfrak{p}|^{(n\alpha+i-1)(s-1/2)}}P_{\mathfrak{p}}(1-s;m).$$

Proposition 4.1. *Define*

$$Z_2'(s,w) = \sum_m \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{L(s,\bar{\chi}_{m_0})Q(s;m)}{|m|^w}.$$

We have $Z_2' = Z_2$.

Proof. Define

$$H_2'(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_i) = \begin{cases} (1-\mathfrak{p}^{-s})^{-1}\sum_{k=0}^{\infty}\frac{Q(s;\mathfrak{p}^{nk})}{\mathfrak{p}^{nkw}} & i=0 \\ \frac{g(1,\epsilon^i,\chi_{\mathfrak{p}})}{\sqrt{|\mathfrak{p}|}}\sum_{k=0}^{\infty}\frac{Q(s;\mathfrak{p}^{nk+i})}{\mathfrak{p}^{nkw}} & 0<i<n. \end{cases} \tag{4.2}$$

Then $H_2'(\mathfrak{p}^{-s},\mathfrak{p}^{-w}) = \sum_{i=0}^{n-1} H_2'(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_i)$ is the $\mathfrak{p}$-part of Z_2'. We will show that H_2' and H_2 both satisfy the functional equations with H_1 shown in Theorem 3.2 and therefore $H_2' = H_2$. The result follows since, for fixed m, the L-functions, P, and Q each have Euler products.

As a result of the definition in equation (4.1), the $\mathfrak{p}$-parts of Q and P satisfy

$$P(s;\mathfrak{p}^{n\alpha+i}) = \begin{cases} \mathfrak{p}^{n\alpha(1/2-s)}Q(1-s;\mathfrak{p}^{n\alpha}) & i=0 \\ \mathfrak{p}^{(n\alpha+i-1)(1/2-s)}Q(1-s;\mathfrak{p}^{n\alpha+i}) & 0<i<n. \end{cases}$$

Therefore, we relate H_1 to H_2' by

$$\begin{aligned} H_1(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_0) =& (1-|\mathfrak{p}|^{-s})^{-1}\sum_{k=0}^{\infty}\frac{P(s;\mathfrak{p}^{nk})}{|\mathfrak{p}|^{nkw}} \\ =& (1-|\mathfrak{p}|^{-s})^{-1}\sum_{k=0}^{\infty}\frac{Q(1-s;\mathfrak{p}^{nk})|\mathfrak{p}|^{nk(1/2-s)}}{|\mathfrak{p}|^{nkw}} \\ =& (1-|\mathfrak{p}|^{-s})^{-1}\sum_{k=0}^{\infty}\frac{Q(1-s;\mathfrak{p}^{nk})}{|\mathfrak{p}|^{nk(w+s-1/2)}} \\ =& \frac{1-|\mathfrak{p}|^{-(1-s)}}{1-|\mathfrak{p}|^{-s}}H_2'(\mathfrak{p}^{-(1-s)},\mathfrak{p}^{-(w+s-1/2)};\delta_0). \end{aligned}$$

This is exactly the functional equation satisfied by H_2 in Theorem 3.2. A similar computation shows that

$$H_1(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_i) = \frac{\sqrt{|\mathfrak{p}|}}{g(1,\epsilon^i,\chi_{\mathfrak{p}})}|\mathfrak{p}|^{s-1/2}H_2(\mathfrak{p}^{-(1-s)},\mathfrak{p}^{-(w+s-1/2)};\delta_i)$$

for $0<i<n$. Thus $H_2'(X,Y;\delta_i) = H_2(X,Y;\delta_i)$ for all $0 \le i < n$ and this completes the proof. □

For $0 \le i < n$, define

$$Z_1(s,w;\delta_i) = \sum_{\substack{m\in\mathcal{O}_{mon} \\ \deg m \equiv i \pmod n}} \frac{L(s,\chi_{m_0})P(s;m)}{|m|^w}$$

and

$$Z_2(s,w;\delta_i) = \sum_{\substack{m\in\mathcal{O}_{mon} \\ \deg m \equiv i \pmod n}} \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{L(s,\bar{\chi}_{m_0})Q(s;m)}{|m|^w}.$$

Theorem 4.2. *We have the functional equation*

$$Z_1(s,w;\delta_i) = \begin{cases} q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}} Z_2(1-s, w+s-\frac{1}{2};\delta_0) & \text{for } i=0 \\ q^{2s-1}q^{1/2-s}\frac{\bar{\tau}(\epsilon^i)}{\sqrt{q}} Z_2(1-s,w+s-\frac{1}{2};\delta_i) & \text{for } 0<i<n. \end{cases}$$

Proof. This is a direct computation utilizing the functional equation (2.5) for $L(s,\chi_{m_0})$. □

5. Convolutions

We define a convolution operation $\star$ on rational functions in x and y with power series expansions around the origin. For

$$A(x,y) = \sum_{j,k\geq 0} a(j,k)x^j y^k \quad \text{and} \quad B(x,y) = \sum_{j,k\geq 0} b(j,k)x^j y^k,$$

define

$$(A\star B)(x,y) = \sum_{j,k\geq 0} a(j,k)b(j,k)x^j y^k.$$

We can compute convolutions as the double integral

$$(A\star B)(x,y) = \left(\frac{1}{2\pi i}\right)^2 \int\int A(u,v)B(\frac{x}{u},\frac{y}{v})\frac{dudv}{uv}, \tag{5.1}$$

where each integral is a counterclockwise circuit of a small circle in the complex plane. (The circle must be small enough that $A(x,y)$ is holomorphic for x,y inside the circle.) We will utilize the residue theorem to compute this contour integral.

6. Evaluation of Z_1 and Z_2

We will now prove Theorem 1.2. We first establish the identity (1.7); then (1.8) will follow from the functional equation (1.1). It follows from Proposition 2.1 that

$$\sum_{\substack{d\in\mathcal{O}_{mon}\\ \deg d=k}} \chi_{m_0}(\hat{d})a(d,m) = 0 \tag{6.1}$$

when $\deg m \leq k$ unless m is a perfect n^{th} power. To prove (1.7) of Theorem 1.2, we begin by writing

$$Z_1(s,w) = Z_a(s,w) + Z_a(w,s) - Z_b(s,w) \tag{6.2}$$

where

$$Z_a(s,w) = \sum_{k\geq j\geq 0} \frac{1}{q^{jw}q^{ks}} \sum_{\substack{d,m\in\mathcal{O}_{mon}\\ \deg m=j\\ \deg d=k}} \chi_{m_0}(\hat{d})a(d,m)$$

and

$$Z_b(s,w) = \sum_{k\geq 0} \frac{1}{q^{kw}q^{ks}} \sum_{\substack{m\in\mathcal{O}_{mon}\\ \deg m=j}} \sum_{\substack{d\in\mathcal{O}_{mon}\\ \deg d=k}} \chi_{m_0}(\hat{d})a(d,m).$$

First, note that

$$\sum_{\substack{m\in\mathcal{O}_{mon}\\ \deg m=j}} \sum_{\substack{d\in\mathcal{O}_{mon}\\ \deg d=k}} \chi_{m_0}(\hat{d})a(d,m) = \sum_{\substack{m\in\mathcal{O}_{mon}\\ \deg m=j}} \sum_{\substack{d\in\mathcal{O}_{mon}\\ \deg d=k}} \chi_{d_0}(\hat{m})a(m,d).$$

When m and d are coprime, the reciprocity law (1.1) guarantees that $\chi_{m_0}(\hat{d}) = \chi_{d_0}(\hat{m})$. Otherwise, when m and d are not coprime, $\chi_{m_0}(\hat{d}) \neq \chi_{d_0}(\hat{m})$ only when there exists a prime $\mathfrak{p}$ such that $\mathfrak{p}|d_0$ and $\mathfrak{p}|m_0$. In this case $a(d,m) = 0$. The symmetry $a(d,m) = a(m,d)$ is obvious. This establishes the validity of the decomposition (6.2) of Z_1.

Now the key observation is that because of equation (6.1), we have

$$Z_a(s,w) = \sum_{k \geq j \geq 0} \frac{1}{q^{jw} q^{ks}} \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = j \\ m_0 = 1}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} a(d,m)$$

and

$$Z_b(s,w) = \sum_{k \geq 0} \frac{1}{q^{kw} q^{ks}} \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = k \\ m_0 = 1}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} a(d,m),$$

that is, the inner sum vanishes unless m is a perfect n^{th}-power. This leads us to consider the series

$$T_a(s,w) = \sum_{\substack{m \in \mathcal{O}_{mon} \\ m_0 = 1}} \sum_{d \in \mathcal{O}_{mon}} \frac{a(d,m)}{|m|^w |d|^s},$$

which has an Euler product

$$\begin{aligned} T_a(s,w) &= \prod_{\mathfrak{p}} \sum_{j,k \geq 0} \frac{a(\mathfrak{p}^j, \mathfrak{p}^{nk})}{|\mathfrak{p}|^{nk} |\mathfrak{p}|^j} \\ &= \prod_{\mathfrak{p}} \frac{1 - |\mathfrak{p}|^{-s-nw}}{(1 - |\mathfrak{p}|^{-s})(1 - |\mathfrak{p}|^{-nw})(1 - |\mathfrak{p}|^{(n-1)-ns-nw})} \\ &= \frac{\zeta(s)\zeta(nw)\zeta(ns + nw - (n-1))}{\zeta(s+nw)} \\ &= \frac{1 - q^{1-s-nw}}{(1 - q^{1-s})(1 - q^{1-nw})(1 - q^{n-ns-nw})} \\ &= \frac{1 - qxy^n}{(1 - qx)(1 - qy^n)(1 - q^n x^n y^n)} \\ &= \tilde{T}_a(x,y), \end{aligned} \tag{6.3}$$

with $x = q^{-s}, y = q^{-w}$.

It is clear that

$$Z_a(s,w) = (\tilde{T}_a \star \tilde{K}_a)(x,y) \tag{6.4}$$

where

$$\begin{aligned} \tilde{K}_a(x,y) &= \frac{1}{(1-x)(1-xy)} \\ &= \sum_{j \geq k \geq 0} x^j y^k. \end{aligned}$$

Compute the convolution in (6.4) by means of the integral in equation (5.1) which we can evaluate using the residue theorem. We find

$$Z_a(s,w) = \frac{1}{(1 - q^{n+1} x^n y^n)(1 - qx)}. \tag{6.5}$$

We can compute Z_b similarly: let $\tilde{K}_b(x,y) = \frac{1}{1-xy}$. Then

$$\begin{aligned} Z_b(s,w) &= (\tilde{T}_a \star \tilde{K}_b)(x,y) \\ &= \frac{1}{1-q^{n+1}x^n y^n}. \end{aligned} \tag{6.6}$$

Putting this all together,

$$\begin{aligned} Z_1(s,w) &= \frac{1}{(1-q^{n+1}x^n y^n)(1-qx)} + \frac{1}{(1-q^{n+1}x^n y^n)(1-qy)} \\ &\quad - \frac{1}{1-q^{n+1}x^n y^n} \\ &= \frac{1-qy+1-qx-(1-qy)(1-qx)}{(1-q^{n+1}x^n y^n)(1-qx)(1-qy)} \\ &= \frac{1-q^2xy}{(1-q^{n+1}x^n y^n)(1-qx)(1-qy)}. \end{aligned} \tag{6.7}$$

This establishes (1.7).

With the rational function for $Z_1(s,w)$, we can use the functional equations relating $Z_1(s,w;\delta_i)$ and $Z_2(s,w;\delta_i)$ for $0 \le i < n$ to evaluate $Z_2(s,w)$. Expanding the geometric series $\frac{1}{1-qy}$ and collecting terms with the exponent on y congruent to $i \pmod n$, we arrive at

$$Z_1(s,w;\delta_i) = \begin{cases} \frac{1-q^{n+1}xy^n}{(1-qx)(1-q^n y^n)(1-q^{n+1}x^n y^n)} & i = 0 \\ \frac{(q^i - q^{i+1}x)y^i}{(1-qx)(1-q^n y^n)(1-q^{n+1}x^n y^n)} & 0 < i < n. \end{cases}$$

Using the functional equations relating $Z_1(s,w,\delta_i)$ and $Z_2(1-s, w+s-\frac{1}{2}, \delta_i)$ and remembering that $\left|\frac{\tau(\epsilon^i)}{\sqrt{q}}\right| = 1$, we see that

$$Z_2(s,w;\delta_i) = \begin{cases} q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}} Z_1(1-s, w+s-\frac{1}{2};\delta_i) & i = 0 \\ q^{2s-1}q^{1/2-s}\frac{\tau(\epsilon^i)}{\sqrt{q}} Z_1(1-s, w+s-\frac{1}{2};\delta_i) & 0 < i < n. \end{cases}$$

With this in hand, Z_2 is

$$q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}} Z_1(1-s, w+s-\frac{1}{2};\delta_0) + \sum_{i=1}^{n-1} q^{2s-1}q^{1/2-s}\frac{\tau(\epsilon^i)}{\sqrt{q}} Z_1(1-s, w+s-\frac{1}{2};\delta_i). \tag{6.8}$$

When simplified, equation (6.8) is the rational function for Z_2 given in Theorem 1.2.

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