MULTIPLE DIRICHLET SERIES OVER RATIONAL FUNCTION FIELDS

GAUTAM CHINTA

Abstract. We explicitly compute some double Dirichlet series constructed from n^{th} order Gauss sums over rational function fields. These turn out to be rational functions in q^{-s_1} and q^{-s_2}, where q is the size of the constant field. Key use is made of the group of 6 functional equations satisfied by these series.

1. Introduction

The purpose of this paper is to explicitly compute some examples of Weyl group multiple Dirichlet series over the rational function field $\mathbb{F}_q(t)$. As described in [2], these are Dirichlet series in r complex variables $s_1, s_2, \ldots, s_r$ whose coefficients can be expressed in terms of n^{th} order Gauss sums. The general theory implies that over a function field, these multiple Dirichlet series will be rational functions of $q^{-s_1}, q^{-s_2}, \ldots, q^{-s_r}$. Except when $n = 2$, no examples of these rational functions have been written down.

Using explicit knowledge of the functional equations, we will express the A_2 series as a rational function of q^{-s_1} and q^{-s_2}. This is the main result of this paper and is given in Theorem 4.2. The functional equations of multiple Dirichlet series arise from the functional equations of single variable Gauss sum Dirichlet series of the type initially studied by Kubota [17] using the theory of metaplectic Eisenstein series on the n-fold cover of GL_2. This theory was further developed by Kazhdan and Patterson [16] who studied Eisenstein series on the n-fold cover of GL_r. It is conjectured that the Weyl group multiple Dirichlet series are related to Whittaker coefficients of these metaplectic Eisenstein series. This conjecture and much supporting evidence for it is given in [2, 4]

Date: 15 Jun 07.

2000 *Mathematics Subject Classification.* Primary 11F66, 11M41; Secondary 11F30, 20F55.

Key words and phrases. Weyl group multiple Dirichlet series, Gauss sums.

In [2] is described a heuristic method to associate to a positive integer n and a root system Φ of rank r, a multiple Dirichlet series Z in r complex variables with coefficients given by n^{th} order Gauss sums. Moreover, Z is expected to have an analytic continuation to $\mathbb{C}^r$ and to satisfy a group of functional equations isomorphic to W, the Weyl group of the root system. Brubaker, Bump, and Friedberg [3] have given a precise definition of Z in the *stable case*; by definition, this means n is sufficiently large for a fixed Φ. In [3] the authors show that for such n, the Weyl group multiple Dirichlet series admit meromorphic continuation and have the expected group of functional equations.

The multiple Dirichlet series studied in this paper fall in the stable range. Therefore, the general shape of the results of this paper is already a consequence of the results of [2, 3]. What is new in this paper is the precise description of the functional equations and the explicitness of Theorem 4.2. The explicit computations in the case of the rational function field are considerably simpler than in the general case. This fact was exploited by J. Hoffstein [15] in his investigations of the theta function θ_n on the n-fold cover of SL_2. We will make use of his results on the Fourier expansions and functional equation of the metaplectic Eisenstein series in this context.

There are two reasons for carrying out the rational function field computation in such detail. First, we believe that the computation of higher rank multiple Dirichlet series can give new information on the Fourier coefficients of theta function θ_n. The nature of the Fourier coefficients of $\theta_n(z)$ for $n \geq 3$ grows increasingly complicated as n increases. Patterson explictly computed $\theta_3(z)$ in [18,19] and formulated a conjecture about the Fourier coefficients of $\theta_4(z)$, see [12]. Despite partial results of Hoffstein [15] and Suzuki [21, 22], the conjecture remains unproven. For $n = 6$, some interesting structure was also noticed by Wellhausen [23]. But for $n = 5$ and $n \geq 7$, there is at present not even a conjectural understanding of the Fourier coefficients of $\theta_n(z)$.

In our work, these mysterious coefficients arise after taking residues in multiple Dirichlet series. Since the multiple Dirichlet series we compute are explicitly given rational functions, one can hope to directly take residues and try to identify the resulting object in terms of known objects. Though the

main focus of this paper are the multiple Dirichlet series associated to the root system A_2, we do also give examples in Section 5 of the cubic ($n = 3$) A_3 series. Taking residues of the cubic A_3 series gives the rational function field analogue of a recent result of Brubaker and Bump [1]. They show that the cubic double Dirichlet series of Friedberg, Hoffstein and Lieman [14] are residues of the cubic A_3 Weyl group multiple Dirichlet series. They interpret their result in terms of the Bump-Hoffstein conjecture [5] and make further conjectures on how the series of Friedberg, Hoffstein and Lieman arise as multiresidues of higher rank multiple Dirichlet series. Unfortunately, their method of proof relies heavily on Patterson's explicit computation of θ_3 and therefore will not readily generalize to higher n. The methods developed in this paper, however, do generalize and hopefully can be used to address these questions.

The second motivation for carrying out these explicit computations is in order to gain insight into the problem of constructing *unstable* Weyl group multiple Dirichlet series. In the unstable case, that is, when n is small relative to Φ, a complete description of the coefficients of the Weyl group multiple Dirichlet series does not yet exist except in the case $n = 2$, which was treated in Chinta and Gunnells [8]. Important partial progress including a beautiful conjecture for multiple Dirichlet series associated to root systems of type A_r is given in Brubaker, Bump, Friedberg and Hoffstein [4]. The conjecture describes the p-parts of the multiple Dirichlet series in terms of Gelfand-Tsetlin patterns. This Gelfand-Tsetlin conjecture is verified to give the correct coefficients in the stable range and also to give the correct coefficients when $n = 2$ for $A_r, r \leq 5$, which were first computed by the author in [6].

The relevance of the present paper to the description of the unstable coefficients is given by the striking resemblance between the rational function field multiple Dirichlet series and the p-part of that series, cf. (3.3) and (4.8). More striking is the resemblance in the A_3 series between the 24 terms of Table 1 of [1] and (5.3). This resemblance was previously noted by the author in the quadratic case ($n = 2$), in which case it can be shown that both the p-part and the multiple Dirichlet series are uniquely characterized

by the functional equations they satisfy. To see this worked out in detail for A_2, we refer the reader to Section 5 of [7].

It is very likely that a similar phenomenon accounts for the resemblance for arbitrary n. Regardless, it suggests a promising approach to the problem of defining unstable Weyl group multiple Dirichlet series–namely, to generalize to arbitrary n, the invariant function methods used in [6, 8, 10] to treat the quadratic case. A first step, using a group action motivated by the functional equation (4.4), has been carried out in a joint work with Gunnells, [9].

This work was supported by NSF Grant FRG DMS-0652605 and by the Alexander von Humboldt Foundation. The author warmly thanks Joel Mohler for very careful readings of drafts of this article and Prof. S. J. Patterson for several illuminating discussions. The author is also grateful to the Hausdorff Research Institute of Mathematics for their hospitality during the completion of this work.

2. Preliminaries

We review some concepts and notation from Patterson, [20]. Let n be an integer ≥ 2 and q a power of an odd prime p. We assume that q is congruent to 1 mod n. For convenience, we also assume that q is congruent to 1 mod 4.

Let $\mu_n = \{a \in \mathbb{F}_q : a^n = 1\}$ and let $\chi : \mathbb{F}_q^\times \to \mu_n$ be the character $a \mapsto a^{\frac{q-1}{n}}$. Let K be the rational function field $\mathbb{F}_q(t)$ with polynomial ring $\mathcal{O} = \mathbb{F}_q[t]$. We let $K_\infty = \mathbb{F}_q((t))$ denote the field of Laurent series in t^{-1}. Also, let $\mathcal{O}_{mon}$ denote the set of monic polynomials in $\mathcal{O}$.

For $x, y \in \mathcal{O}$ relatively prime, $\left(\frac{x}{y}\right)$ denotes the n^{th} order power residue symbol. We have the reciprocity law

$$\left(\frac{x}{y}\right) = \left(\frac{y}{x}\right) \tag{2.1}$$

for x, y monic. (Here we make use of the fact that q is congruent to 1 mod 4.)

We next define an additive character on K_∞. First let e_0 be a nontrivial additive character on $\mathbb{F}_p$. Use this to define a character $e_\star$ of $\mathbb{F}_q$ by $e_\star(a) = e_0(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} a)$. Let ω be the global differential dx/x^2. Finally define

the character e of K_∞ by $e(y) = e_\star(\mathrm{Res}_\infty(\omega y))$ for $y \in K_\infty$. Note that

$$\{y \in K : e|y\mathcal{O} = 1\} = \mathcal{O}.$$

Fix an embedding ϵ from the the n^{th} roots of unity of $\mathbb{F}_q$ to $\mathbb{C}^\times$. For $r, c \in \mathcal{O}$ we define the Gauss sum

$$g(r, \epsilon, c) = \sum_{y \mod c} \epsilon\left(\left(\frac{y}{c}\right)\right) e\left(\frac{ry}{c}\right).$$

The main subject of this paper is Dirichlet series and multiple Dirichlet series constructed from such Gauss sums. It will be necessary for us to consider sums over certain ideal classes in $\mathcal{O}$. To this end, for $x, y \in K_\infty$ we write $x \sim y$ if $x/y \in K_\infty^{\times n}$.

Define the Dirichlet series

$$\psi(r, \epsilon, \eta, s) = (1 - q^{n-ns})^{-1} \sum_{\substack{c \in \mathcal{O}_{mon} \\ c \sim \eta}} g(r, \epsilon, c)|c|^{-s}$$

where the sum is over all nonzero monic polynomials $c \sim \eta$ and $|c|$ is $q^{\deg c}$. The η we will use are of the form π_∞^{-i}, $0 \le i < n$. We will henceforth suppress the embedding ϵ from the notation, and identify the value of a power residue symbol with its image in $\mathbb{C}$ under ϵ. Thus $\psi(r, \pi_\infty^{-i}, s) = \psi(r, \epsilon, \pi_\infty^{-i}, s)$ and $g(r, c) = g(r, \epsilon, c)$. We also allow linear combinations of the π_∞^i's. So, for example, letting $\mathbb{I} = \sum_{i=0}^{n-1} \pi_\infty^{-i}$, we have

$$\psi(r, \mathbb{I}, s) = \sum_{i=0}^{n-1} \psi(r, \pi_\infty^{-i}, s) = \sum_{c \in \mathcal{O}_{mon}} g(r, c)|c|^{-s}.$$

We now describe the functional equation which the Gauss sum Dirichlet series $\psi(r, \pi_\infty^{-i}, s)$ satisfies. Let i, j be integers mod n and r monic of degree $d = nk + j, n \ge 0$. Define

$$P_{ij}(s) = P_{i,\deg r}(s) = -q^{(1-s)(1-(j+1-2i)_n)} \frac{q-1}{1 - q^{n+1}q^{-ns}}$$

and

$$Q_{ij}(s) = Q_{i,\deg r}(s) = -\tau(\epsilon^{2i-j-1}) q^{(1-s)(1-n)} \frac{1 - q^n q^{-ns}}{1 - q^{n+1}q^{-ns}}.$$

Here, $(\beta)_n = \beta - n\lfloor \beta/n \rfloor$ and $\tau(\epsilon^j)$ is the Gauss sum

$$\tau(\epsilon^j) = \sum_{a \in \mathbb{F}_q^\times} \epsilon^j(\chi(a)) e_\star(a).$$

Then it is proven in [15, Proposition 2.1] that ψ satisfies the functional equation

$$\psi(r,\pi_\infty^{-i},s) = |r|^{1-s}P_{i,\deg r}(s)\psi(r,\pi_\infty^{-i},2-s) + |r|^{1-s}Q_{i,\deg r}(s)\psi(r,\pi_\infty^{i-\deg r-1},2-s). \tag{2.2}$$

(See also [20, Eq. 2.2].) Summing both sides over i, we can also write this as

$$\psi(r,\mathbb{I},s) = |r|^{1-s}\sum_{i=0}^{n-1} T_{i,\deg r}(s)\psi(r,\pi_\infty^{-i},2-s) \tag{2.3}$$

where, if $2i-j-1 \not\equiv 0 \pmod n$,

$$T_{ij}(s) = -\frac{q^{(1-s)(1-n)}}{1-q^{n+1-ns}}\left[q^{(1-s)(2i-j-1)_n}(q-1) + \tau(\epsilon^{2i-j-1})(1-q^{n(1-s)})\right]$$

and if $2i-j-1 \equiv 0 \pmod n$,

$$T_{ij}(s) = q^{(1-n)(1-s)}.$$

We observe that for fixed s, each of the functions P_{ij}, Q_{ij}, T_{ij} depends only on $2i-j$. This fact will be needed later.

The general theory [15,16] tells us that $(1-q^{n+1-ns})\psi(r,\pi_\infty^{-i},s)$ is a polynomial in q^{-s}. The functional equation then allows us to give a bound on the degree of this polynomial, see e.g. [15, Prop. 2.1] or [20]. As a simple consequence we have

Proposition 2.1. *The Gauss sum Dirichlet series associated to the constant polynomial is*

$$\psi(1,\mathbb{I},s) = \frac{1+q\tau(\epsilon)}{1-q^{n+1-ns}}.$$

The main subject of the papers of Hoffstein [15] and Patterson [20] are the residues of the Gauss sum Dirichlet series $\psi(r,\mathbb{I},s)$ at $s = 1+1/n$. These residues are related to the Fourier coefficients of the theta function on the n-fold metaplectic cover of $GL_2(K)$. Following [20] we define

$$\rho_n(r) = \lim_{s\to 1+\frac{1}{n}}(1-q^{n+1-ns})\psi(r,\mathbb{I},s).$$

In Section 5 we indicate how the theory of multiple Dirichlet series can be used to deduce information on the coefficients $\rho(r)$.

3. The A_2 multiple Dirichlet series

In this section we define and describe the functional equations of a double Dirichlet series constructed from n^{th} order Gauss sums. This series is heuristically of the form

$$\sum_{c_1}\sum_{c_2}\frac{g(1,\epsilon,c_1)g(1,\epsilon,c_2)\left(\frac{c_1}{c_2}\right)^{-1}}{|c_1|^{s_1}|c_2|^{s_2}},$$

where the sum is over all c_1, c_2 nonzero monic polynomials.

More precisely, we define

$$Z(s_1,s_2;\eta_1,\eta_2) = \tag{3.1}$$
$$(1-q^{n-ns_1})^{-1}(1-q^{n-ns_2})^{-1}(1-q^{2n-ns_1-ns_2})^{-1}\sum_{c_1\sim\eta_1}\sum_{c_2\sim\eta_2}\frac{H(c_1,c_2)}{|c_1|^{s_1}|c_2|^{s_2}},$$

where the coefficient $H(c_1,c_2)$ is defined by the following two conditions:

(1) If $gcd(c_1c_2,d_1d_2)=1$ then

$$\frac{H(c_1d_1,c_2d_2)}{H(c_1,c_2)H(d_1,d_2)} = \left(\frac{c_1}{d_1}\right)\left(\frac{d_1}{c_1}\right)\left(\frac{c_2}{d_2}\right)\left(\frac{d_2}{c_2}\right)\left(\frac{c_1}{d_2}\right)^{-1}\left(\frac{d_1}{c_2}\right)^{-1}. \tag{3.2}$$

(2) If p is prime, then

$$\sum_{k,l\geq 0}H(p^k,p^l)x^ky^l = 1+g(1,p)x+g(1,p)y+g(1,p)g(p,p^2)xy^2 \tag{3.3}$$
$$+g(1,p)g(p,p^2)x^2y+g(1,p)^2g(p,p^2)x^2y^2.$$

It can be seen that summing (3.1) over one of the indices, say c_1, with the other index fixed will produce a Dirichlet series

$$E(c_2,\eta,s_1) = (1-q^{n-ns_1})^{-1}\sum_{\substack{c_1\in\mathcal{O}_{mon}\\ c_1\sim\eta}}H(c_1,c_2)|c_1|^{-s_1} \tag{3.4}$$

which is closely related to a Gauss sum Dirichlet series. This will have a functional equation as $s_1\mapsto 2-s_1$, which will in turn induce a functional equation in the double Dirichlet series Z. The rest of this section is devoted to verifying this assertion and describing the precise functional equation of Z.

Let a be a function defined on $\mathbb{F}_q[t]$. We say that a is *twisted multiplicative* if $a(xy) = a(x)a(y)\left(\frac{x}{y}\right)\left(\frac{y}{x}\right)$ whenever x and y are relatively prime. The first Lemma below is a standard property of Gauss sums.

Lemma 3.1. *Let r be a monic polynomial in $\mathbb{F}_q[t]$. The map*

$$x \mapsto g(r, x)$$

is twisted multiplicative.

Lemma 3.2. *Fix a cubefree monic polynomial c. Write $c = c_1c_2^2$ where c_1 is monic and squarefree. The map*

$$x \mapsto \frac{H(xc_2, c)}{H(c_2, c)}$$

is twisted multiplicative.

Proof of Lemma 3.2. First note that c_2 is the minimal monic polynomial such that $H(c_2, c) \neq 0$, i.e., if $H(d, c) \neq 0$ then $c_2|d$. To prove the Lemma, take relatively prime monic polynomials x and y. We need to show

$$H(xyc_2, c) = \frac{H(xc_2, c)H(yc_2, c)}{H(c_2, c)}\left(\frac{x}{y}\right)\left(\frac{y}{x}\right) \tag{3.5}$$

Write

$$c_2 = c_2^{(1)}c_2^{(2)}, c = c^{(1)}c^{(2)}, \text{ with } (xc_2^{(1)}c^{(1)}, yc_2^{(2)}c^{(2)}) = 1$$

and compute both sides of (3.5) using (3.2). □

We can now express $E(c, \eta, s)$ in terms of a Gauss sum Dirichlet series.

Lemma 3.3. *Let c be a cubefree monic polynomial and write $c = c_1c_2^2$ as above. Then*

$$E(c, \mathbb{I}, s) = \frac{H(c_2, c)}{|c_2|^s}\psi(c_1, \mathbb{I}, s).$$

Proof. Having already established the twisted multiplicativity of the coefficients

$$x \mapsto \frac{H(c_2x, c)}{H(c_2, c)}$$

it remains only to verify that

$$\frac{H(c_2P^l, c)}{H(c_2, c)} = g(c_1, P^l) \tag{3.6}$$

for all irreducible polynomials P and integers $l \geq 0$. Note that

$$H(c_2, c) = H(1, c_1)H(c_2, c_2^2)\left(\frac{c_1}{c_2^2}\right)\left(\frac{c_2^2}{c_1}\right)\left(\frac{c_2}{c_1}\right)^{-1}. \tag{3.7}$$

To go further, we break the argument into three cases, depending on whether $(P, c) = 1, P||c_1$ or $P||c_2$. Since the equality is trivially satisfied for $l = 0$, we will assume that $l > 0$.

If $(P, c) = 1$ the numerator of the left hand side of (3.6) is

$$H(c_2, c)H(P^l, 1)\left(\frac{c_2}{P^l}\right)\left(\frac{P^l}{c_2}\right)\left(\frac{P^l}{c}\right)^{-1}.$$

This is nonzero only when $l = 1$, in which we get

$$H(c_2P, c) = H(c_2, c)g(c_1, P)$$

as desired.

If $P|c_1$ we compute

$$\begin{aligned} &H(P^l \cdot c_2, c_1 \cdot c_2^2) \\ &= H(P^l, c_1)H(c_2, c_2^2)\left(\frac{P^l}{c_2}\right)\left(\frac{c_2}{P^l}\right)\left(\frac{c_1}{c_2^2}\right)\left(\frac{c_2^2}{c_1}\right)\left(\frac{P^l}{c_2^2}\right)^{-1}\left(\frac{c_2}{c_1}\right)^{-1} \end{aligned} \tag{3.8}$$

Since $P||c_1$ this expression is nonzero only when $l = 2$. Write $c_1 = P\hat{c}_1$. Thus $H(1, c_1) = H(1, \hat{c}_1)H(1, P)\left(\frac{P}{\hat{c}_1}\right)\left(\frac{\hat{c}_1}{P}\right)$. Using this and again the definition (3.2),(3.3) of H, we find

$$\begin{aligned} H(P^2, c_1) &= H(P^2, P)H(1, P)^{-1}H(1, c_1)\left(\frac{P^2}{\hat{c}_1}\right)^{-1} \\ &= g(P, P^2)H(1, c_1)\left(\frac{P^2}{\hat{c}_1}\right)^{-1} \\ &= g(c_1, P^2)H(1, c_1). \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.7) with (3.9) we conclude that

$$\frac{H(c_2P^2, c)}{H(c_2, c)} = g(c_1, p^2),$$

as was to be shown.

The proof of the third case (when $P|c_2$) is similar and will be omitted. □

Lemma 3.4. *Let c be a monic polynomial. Then*

$$E(c,\mathbb{I},s)=|c|^{1-s}\sum_{i=0}^{n-1}T_{i,\deg c}(s)E(c,\pi_\infty^{-i},2-s).$$

Equivalently,

$$E(c,\pi_\infty^{-i},s)=$$
$$|c|^{1-s}\left(P_{i,\deg c}(s)E(c,\pi_\infty^{-i},2-s)+Q_{i,\deg c}(s)E(c,\pi_\infty^{i-\deg c-1},2-s)\right).$$

The functions P_{ij},Q_{ij},T_{ij} are as defined at the end of Section 2.

Proof. We may assume c is cubefree, as otherwise $D(c,s)=0$. Write $c=c_1c_2^2$ with c_1 monic and squarefree. By the previous Lemma,

$$E(c,\mathbb{I},s)=\frac{H(c_2,c)}{|c_2|^s}\psi(c_1,\mathbb{I},s).$$

Let $\deg c_k\equiv j_k(n),\ 0\le j_k<n$ for $k=1,2$. Thus

$$\begin{aligned}
E(c,\mathbb{I},s) &= \frac{H(c_2,c)}{|c_2|^s}\psi(c_1,\mathbb{I},s)\\
&= H(c_2,c)\sum_{i=0}^{n-1}|c_1|^{1-s}T_{ij_1}(s)\frac{\psi(c_1,\pi_\infty^{-i},2-s)}{|c_2|^s}\\
&= H(c_2,c)\sum_{i=0}^{n-1}|c|^{1-s}T_{ij_1}(s)\frac{\psi(c_1,\pi_\infty^{-i},2-s)}{|c_2|^{2-s}}\\
&= |c|^{1-s}\sum_{i=0}^{n-1}T_{ij_1}(s)E(c,\pi_\infty^{-i-j_2},2-s)\\
&= |c|^{1-s}\sum_{i=0}^{n-1}T_{i-j_2,j_1}(s)E(c,\pi_\infty^{-i},2-s).
\end{aligned}$$

The proof is completed by noting that $T_{i-j_2,j_1}(s)=T_{i,j_1+2j_2}(s)=T_{i,\deg c}(s)$. □

From Lemma 3.4 follow immediately the functional equations of $Z(s_1,s_2)$.

Theorem 3.5. *Let $0\le i,j<n$. The collection of double Dirichlet series $Z(s_1,s_2;\pi_\infty^{-i},\pi_\infty^{-j})$ satisfy the functional equations*

$$\begin{aligned}
(3.10)\quad Z(s_1,s_2;\pi_\infty^{-i},\pi_\infty^{-j}) &= P_{ij}(s_1)Z(2-s_1,s_1+s_2-1;\pi_\infty^{-i},\pi_\infty^{-j})\\
&\quad+Q_{ij}(s_1)Z(2-s_1,s_1+s_2-1;\pi_\infty^{-j-1+i},\pi_\infty^{-j})
\end{aligned}$$

and

$$Z(s_1,s_2;\pi_\infty^{-i},\pi_\infty^{-j}) = P_{ji}(s_2)Z(s_1+s_2-1,2-s_2;\pi_\infty^{-i},\pi_\infty^{-j}) + Q_{ji}(s_2)Z(s_1+s_2-1,2-s_2;\pi_\infty^{-i},\pi_\infty^{-i-1+j}). \tag{3.11}$$

Remark. Let G be the symmetric group on three letter with generating reflections σ_1,σ_2. Let V be the set of double Dirichlet series with meromorphic continuation to $\mathbb{C}^2$. For $f(s_1,s_2)=f(s_1,s_2;\mathbb{I},\mathbb{I})$ in V, define

$$(f|\sigma_1)(s_1,s_2) = \sum_{i,j=0}^{n} \big(P_{ij}(s_1)f(2-s_1,s_1+s_2-1;\pi_\infty^{-i},\pi_\infty^{-j}) + Q_{ij}(s_1)f(2-s_1,s_1+s_2-1;\pi_\infty^{-j-1+i},\pi_\infty^{-j})\big)$$

and

$$(f|\sigma_2)(s_1,s_2) = \sum_{i,j=0}^{n} \big(P_{ji}(s_2)f(s_1+s_2-1,2-s_2;\pi_\infty^{-i},\pi_\infty^{-j}) + Q_{ji}(s_2)\ f(s_1+s_2-1,2-s_2;\pi_\infty^{-i},\pi_\infty^{-i-1+j})\big)\,.$$

It turns out that these two transformations generate an action of G on V. The functional equations of the previous theorem assert the invariance of $Z(s_1,s_2)$ under this group action.

4. Determination of the A_2 multiple Dirichlet series

In this section we will explicitly write down the double Dirichlet series of the previous section as rational functions in q^{-s_1},q^{-s_2}. We will find it convenient to introduce the variables $x=q^{-s_1},y=q^{-s_2}$. We write

$$\mathcal{P}_{ij}(x) = -(qx)^{1-(-2i+j+1)_n}\frac{q-1}{1-q^{n+1}x^n} \tag{4.1}$$

$$\mathcal{Q}_{ij}(x) = -\tau(\epsilon^{2i-j-1})(qx)^{1-n}\frac{1-q^nx^n}{1-q^{n+1}x^n}, \tag{4.2}$$

and $\mathcal{T}_{ij}(x) =$

$$\begin{cases} -(qx)^{1-n}\frac{(qx)^{(2i-j-1)_n}(q-1)+\tau(\epsilon^{2i-j-1})(1-q^nx^n)}{1-q^{n+1}x^n} & \text{if } 2i-j-1\not\equiv 0 \\ (qx)^{1-n} & \text{otherwise} \end{cases} \tag{4.3}$$

We also introduce

$$\mathcal{Z}(x,y;i,j) = Z(s_1,s_2;\pi_\infty^{-i},\pi_\infty^{-j}) \text{ and } \mathcal{Z}(x,y;\mathbb{I},\mathbb{I}) = \sum_{0\leq i,j<n} \mathcal{Z}(x,y;i,j).$$

When we wish to make the notation reflect the dependence on n, we shall write $Z^{(n)}(x,y;i,j)$.

Then the functional equation (3.10) takes the form

(4.4)
$$\mathcal{Z}(x,y;i,j) \;=\; \mathcal{P}_{ij}(x)\mathcal{Z}(\tfrac{1}{q^2x},qxy;i,j) + \mathcal{Q}_{ij}(x)\mathcal{Z}(\tfrac{1}{q^2x},qxy;j+1-i,j).$$

Summing over i,j between 0 and $n-1$ we can also write this as

$$\mathcal{Z}(x,y,\mathbb{I},\mathbb{I}) = \sum\sum \mathcal{T}_{ij}(x)\mathcal{Z}(\tfrac{1}{q^2x},qxy;i,j). \tag{4.5}$$

Because of the reciprocity law we have as well the relation

$$\mathcal{Z}(x,y;\mathbb{I},\mathbb{I}) = \mathcal{Z}(y,x;\mathbb{I},\mathbb{I}). \tag{4.6}$$

Knowledge of these functional equations allows the explicit computation of $Z(x,y;\mathbb{I},\mathbb{I})$. Let $D(x,y) = (1-q^{n+1}x^n)(1-q^{n+1}y^n)(1-q^{2n+1}x^ny^n)$ and set $N(x,y) = D(x,y)\mathcal{Z}(x,y;\mathbb{I},\mathbb{I})$. The following proposition shows that $D(x,y)$ is the denominator of $\mathcal{Z}$ and gives a bound on the degree of the numerator.

Proposition 4.1. *The function $N(x,y)$ is a polynomial of x,y of degree bounded by 2 in both variables.*

Proof. The fact that the product $N(x,y) = (1-q^{n+1}x^n)(1-q^{n+1}y^n)(1-q^{2n+1}x^ny^n)\mathcal{Z}(x,y;\mathbb{I},\mathbb{I})$ is entire is identical to the proof of Theorem 2 in [2]. To show that $N(x,y)$ is a polynomial and bound the degrees, we argue as in the proof of Theorem 4.1 of Fisher-Friedberg [13]. Let $\vec{\mathcal{Z}}(x,y)$ denote the column vector consisting of all the $\mathcal{Z}(x,y;i,j)$, with $0\leq i,j<n$ (with the pairs (i,j) in some fixed order). In matrix notation the functional equation (4.4) can be expressed as

$$\vec{\mathcal{Z}}(x,y) = A(x)\vec{\mathcal{Z}}(\tfrac{1}{q^2x},qxy)$$

where $A(x)$ is an $n^2\times n^2$ matrix whose coefficients are the functions $\mathcal{P}_{ij}(x)$ and $\mathcal{Q}_{ij}(x)$. From (4.1) we have $A(x) << x^{1-n}$, that is, every entry of the matrix $A(x)$ satisfies this bound as $x\to\infty$. Similarly,(4.6) implies

$$\vec{\mathcal{Z}}(x,y) = B\vec{\mathcal{Z}}(y,x)$$

where B is the identity matrix. (We write it like this because it helps us keep track of an application of the functional equation.) Repeatedly applying the two functional equations, we get

$$\vec{\mathcal{Z}}(x,y) = A(x)BA(qxy)BA(y)B\vec{\mathcal{Z}}(\tfrac{1}{q^2x},\tfrac{1}{q^2y}).$$

Multiply both sides by $D(x,y)D(\frac{1}{q^2x},\frac{1}{q^2y})$:

$$D(\tfrac{1}{q^2x},\tfrac{1}{q^2y})\vec{N}(x,y) = D(x,y)A(x)BA(qxy)BA(y)B\vec{N}(\tfrac{1}{q^2x},\tfrac{1}{q^2y}) \tag{4.7}$$

where $\vec{N}(x,y)$ is the vector with components $D(x,y)\mathcal{Z}(x,y;i,j)$. To show that $N(x,y)$ is a polynomial of the stated degree, it suffices to show that that each entry of $\vec{N}(x,y)$ is $O(|xy|^2)$. Let $x,y \to \infty$ in (4.7). The terms $D(1/(q^2x),1/(q^2y))$ and $\vec{N}(1/(q^2x),1/(q^2y))$ remain bounded, while

$$D(x,y) = O(|xy|^{2n}) \text{ and } \ A(x)A(y)A(xy) = O(|xy|^{2-2n}).$$

Therefore the right hand side is $O(x^2y^2)$. This establishes that $N(x,y)$ is a polynomial in x and y of degree at most 2 in both x and y. □

We now present our main result.

Theorem 4.2. *For $n > 2$ we have*

$$Z^{(n)}(x,y;\mathbb{I},\mathbb{I}) = \frac{1+\tau_1 q\,x+\tau_1 q\,y+\tau_1\tau_2 q^3\,x^2\,y+\tau_1\tau_2 q^3 xy^2+\tau_1^2\tau_2 q^4\,x^2\,y^2}{(1-q^{n+1}x^n)(1-q^{n+1}y^n)(1-q^{2n+1}x^ny^n)} \tag{4.8}$$

where $\tau_i = \tau(\epsilon^i)$.

Remark 4.3. The case $n = 2$ is dealt with in Fisher-Friedberg [13] and Chinta-Friedberg-Hoffstein [7], where it is shown that

$$Z^{(2)}(x,y;\mathbb{I},\mathbb{I}) = \frac{1+q^{3/2}x+q^{3/2}y-q^{9/2}x^2y-q^{9/2}xy^2-q^6x^2\,y^2}{(1-q^3x^2)(1-q^3y^2)(1-q^5x^2y^2)}.$$

The method of proof below will work in this case as well, but a slight adjustment needs to be made to deal with a degenerate Gauss sum.

Proof of Theorem 4.2. We first show that $\mathcal{Z}(x,y;1,1) = 0$. With $i = j = 1$, the functional equation (4.4) implies that

$$\mathcal{Z}(x,y;1,1) = [\mathcal{P}_{11}(x) + \mathcal{Q}_{11}(x)]\,\mathcal{Z}(\tfrac{1}{q^2x},qxy;1,1).$$

Multplying through by $D(x,y)$, we get

$$N(x,y;1,1) = -x\frac{1-q^{n+1}x^n}{1-q^{n-1}x^n}N\left(\frac{1}{q^2x},qxy;1,1\right).$$

But because of Proposition 4.1 we know that $N(x,y;1,1)$ is a constant multiple of the monomial xy. It follows that $N(x,y;1,1)=0$.

Furthermore, setting $x=0$ and $y=0$ and using Proposition 2.1, we deduce that $N(x,y)$ is of the form

$$N(x,y) = 1+q\tau(\epsilon)x+q\tau(\epsilon)y+\alpha x^2y+\alpha xy^2+\beta x^2y^2,$$

for some constants α,β. To determine these constants, we again use the functional equations. For example,

$$\mathcal{Z}(x,y;2,1) = \mathcal{P}_{21}(x)\mathcal{Z}(\tfrac{1}{q^2x},qxy;2,1)+\mathcal{Q}_{21}(x)\mathcal{Z}(\tfrac{1}{q^2x},qxy;0,1).$$

We solve for α and find $\alpha = q^3\tau(\epsilon)\tau(\epsilon^2)$. Similarly applying the functional equation to $\mathcal{Z}(x,y;2,2)$ we find $\beta=\tau_1^2\tau_2q^4$. This completes the proof of the theorem. □

5. Examples

In this section we compute some examples of residues of multiple Dirichlet series and deduce information on the residues ρ_n of the Gauss sum Dirichlet series. By Lemma 3.3, we have

$$\lim_{s_1\to 1+\frac{1}{n}}(1-q^{n+1-ns_1})Z(s_1,s_2;\mathbb{I},\mathbb{I}) = \sum_{\substack{c_1,c_2\in\mathcal{O}_{mon}\\(c_1,c_2)=1\\squarefree}}\frac{H(c_2,c_1c_2^2)\rho_n(c_1)}{|c_1c_2^2|^{s_2}|c_2|^{2+2/n}}. \tag{5.1}$$

When $n=3$, we have

$$Z(x,y;\mathbb{I},\mathbb{I}) = \frac{1+q\tau x+q\tau y+q^4x^2y+q^4xy^2+q^5\tau x^2y^2}{(1-q^4x^3)(1-q^4y^3)(1-q^7x^3y^3)}$$

where $\tau=\tau(\epsilon)$. Note that this series has a simple pole at $s_2=4/3$. The residue of Z at the simple pole $s_1=4/3$ is a constant multiple of

$$\frac{1+\tau q^{-1/3}}{(1-q^{4/3}y)(1-q^3y^3)} = (const)\zeta(s_2-1/3)\zeta(3s_2-2).$$

Comparison of this Dirichlet series with (5.1) suggests that, for a squarefree monic polynomial c,

$$\rho_3(c) = (const)\overline{g(1,c)}|c|^{-2/3}.$$

This agrees with [18], [19], in which Patterson determines the Fourier coefficients of the cubic theta function.

Our second example involves the A_3 series constructed from cubic Gauss sums. We will not provide complete details here as we plan to return to the topic more systematically in a later work. Let

$$\Phi(s_1,s_2,s_3) = (1-q^{n-ns_1})^{-1}(1-q^{n-ns_2})^{-1}(1-q^{2n-ns_1-ns_2})^{-1} \\ (1-q^{n-ns_3})^{-1}(1-q^{2n-ns_2-ns_3})^{-1}(1-q^{3n-ns_1-ns_2-ns_3})^{-1}.$$

The A_3 series is defined by

$$Z(s_1,s_2,s_3) = \Phi(s_1,s_2,s_3)\sum_{c_1,c_2,c_3\in\mathcal{O}_{mon}}\frac{H(c_1,c_2,c_3)}{|c_1|^{s_1}|c_2|^{s_2}|c_3|^{s_3}} \tag{5.2}$$

where the coefficient $H(c_1,c_2,c_3)$ satisfies the twisted multiplicativity Eq. (16) of [1] and is defined in Table 1 of [1] for prime power arguments

When $n=3$, a lengthy computation similar to that given in the proof of Theorem 4.2 shows that $Z(s_1,s_2,s_3)$ is a rational function in $x=q^{-s_1}, y=q^{-s_2}, z=q^{-s_3}$ with denominator

$$(1-q^4x^3)(1-q^4y^3)(1-q^4z^3)(1-q^7x^3y^3)(1-q^7y^3z^3)(1-q^{10}x^3y^3z^3)$$

and numerator the sum of 24 terms

$$\begin{aligned}1 &+ q\tau x + q\tau y + q\tau z + q^4x^2y + q^4xy^2 + q^4y^2z + q^4yz^2 + q^5\tau x^2y^2 \\ &+ q^5\tau y^2z^2 + q^2\tau^2xz - q^{12}x^3y^3z^3 - q^9\tau xy^3z^3 - q^9\tau x^3y^2z^2 - q^9\tau x^2y^2z^3 \\ &- q^9\tau x^3y^3z - q^8xy^2z^3 - q^8x^3y^2z - q^{12}x^2y^4z^3 - q^{12}x^3y^4z^2 - q^6\tau^2xy^3z \\ &+ q^6\bar{\tau}x^2yz^2 - q^{10}\bar{\tau}x^2y^4z^2 - q^{13}\tau x^3y^4z^3.\end{aligned} \tag{5.3}$$

We now show that residues of this series give the the rational function field analogues of the two multiple Dirichlet series considered by Friedberg-Hoffstein-Lieman [14]. These analogues have been defined and computed

by Chinta and Mohler, [11]. They are given by

$$Z_{FHL,1}(s,w) = \tag{5.4}$$
$$(1-q^{1-3s})^{-1}(1-q^{1-3w})^{-1}(1-q^{3-3s-3w})^{-1}\sum_{c_1,c_2\in\mathcal{O}_{mon}}\frac{\left(\frac{c_1}{c_2}\right)a(c_1,c_2)}{|c_1|^s|c_2|^w}$$

and

$$Z_{FHL,2}(s,w) = (1-q^{n/2-ns})^{-1}\sum_{c_1,c_2\in\mathcal{O}_{mon}}\frac{g(c_2,c_1)}{|c_1|^{s+1/2}|c_2|^w}. \tag{5.5}$$

Here $a(g,f)$ is a multiplicative weighting factor defined on prime powers by

$$\sum_{k,l\geq 0}a(\mathfrak{p}^k,\mathfrak{p}^l)x^ky^l = 1+x+y+x^2+y^2-x^3y-xy^3-y^2x^3-x^2y^3-x^3y^3.$$

(The shift by $1/2$ in the s variable in $Z_{FHL,2}$ occurs because the Gauss sums of [14] are normalized to have modulus 1. Also we have reversed the s and w.)

It is shown in [11] that

$$Z_{FHL,1}(s,w) = \frac{1-q^{2-s-w}}{(1-q^{1-s})(1-q^{1-w})(1-q^{4-3s-3w})}$$

and

$$Z_{FHL,2}(s,w) = \frac{1+q^{1/2-s}\tau+q^{3/2-s-w}\tau+q^{2-2s-2}\bar{\tau}-q^{3-2s-2w}\bar{\tau}-q^{9/2-3s-2w}}{(1-q^{1-w})(1-q^{5/2-3s})(1-q^{9/2-3s-3w})}.$$

On the other hand,

$$\lim_{s_3\to 4/3}(1-q^{4-3s_3})Z(s_1,s_2,s_3) = \tag{5.6}$$
$$\frac{(1+q^{-1/3}\tau)(1+q\tau x-q^{7/3}\tau xy+q^{10/3}\bar{\tau}x^2y-q^{14/3}\bar{\tau}x^2y^2-q^{20/3}x^3y^2)}{(1-q^{4/3}y)(1-q^4x^3)(1-q^7x^3y^3)(1-q^3y^3)(1-q^6x^3y^3)}.$$

Hence

$$\operatorname*{Res}_{s_3=4/3}Z(s_1,s_2,s_3) =$$
$$(const)\zeta(3s_2-2)\zeta(3s_1+3s_2-5)Z_{FHL,2}(s_1-1/2,s_2-1/3).$$

Similarly,

$$\operatorname*{Res}_{s_2=4/3}Z(s_1,s_2,s_3) = (const)\zeta(3s_1-2)\zeta(3s_3-2)Z_{FHL,1}(s_1-1/3,s_3-1/3).$$

This identity of the residues of the cubic A_3 multiple Dirichlet series with the series studied in [14] was first established by Brubaker and Bump, [1]. (Actually, Brubaker and Bump work over the number field $\mathbb{Q}(\sqrt{-3})$ but, up to a finite number of bad primes, their methods will work over any global field containing a cube root of unity.)

Guided by the Bump-Hoffstein conjecture [5], Brubaker and Bump further suggest that $n-2$-fold residues of the n^{th} order Weyl group multiple Dirichlet series associated to the root system A_n should coincide with the n^{th} order double Dirichlet series of Friedberg-Hoffstein-Lieman. We hope that explicit computations over the rational function field such as those described above will give more evidence for this expectation.

References

[1] B. Brubaker and D. Bump, *Residues of Weyl Group Multiple Dirichlet Series Associated to GL(n + 1),* In Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proc. Sympos. Pure Math., 75, Amer. Math. Soc., Providence, RI, 2006.

[2] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein, *Weyl group multiple Dirichlet series I*, In Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proc. Sympos. Pure Math., 75, Amer. Math. Soc., Providence, RI, 2006.

[3] B. Brubaker, D. Bump, and S. Friedberg, *Weyl group multiple Dirichlet series II: The stable case*, Invent. Math. 165 (2006), no. 2, 325–355.

[4] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein, *Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_r*, to appear in Annals of Mathematics.

[5] D. Bump and J. Hoffstein, *Some conjectured relationships between theta functions and Eisenstein series on the metaplectic group*, Number theory (New York, 1985/1988), 1–11, Lecture Notes in Math., 1383, Springer, Berlin, 1989.

[6] G. Chinta, *Mean values of biquadratic zeta functions*, Invent. Math. **160** (2005), 145–163.

[7] G. Chinta, S. Friedberg and J. Hoffstein, *Multiple Dirichlet Series and Automorphic Forms*, In Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proc. Sympos. Pure Math., 75, Amer. Math. Soc., Providence, RI, 2006.

[8] G. Chinta and P. Gunnells, *Weyl group multiple Dirichlet series constructed from quadratic characters,* Invent. Math. 167 (2007), no.2, 327-353.

[9] G. Chinta and P. Gunnells, *Weyl group multiple Dirichlet series of type A_2,* preprint.

[10] G. Chinta, S.Friedberg and P. Gunnells, *On the p-parts of quadratic Weyl group multiple Dirichlet series*, to appear in J. Reine Angew. Math.

[11] G. Chinta and J. Mohler, *Sums of L-functions over the rational function field*, preprint.

[12] C. Eckhardt and S.J. Patterson, *On the Fourier coefficients of biquadratic theta series*, Proc. London Math. Soc. (3) 64 (1992), no. 2, 225–264.

[13] B. Fisher and S. Friedberg, *Double Dirichlet series over function fields*, Compos. Math. **140** (2004), no. 3, 613–630.
[14] S. Friedberg, J. Hoffstein, and D. Lieman, *Double Dirichlet series and the n-th order twists of Hecke L-series*, Math. Ann. **327** (2003), no. 2, 315–338.
[15] J. Hoffstein, *Theta functions on the n-fold metaplectic cover of* SL(2)*—the function field case*, Invent. Math. 107 (1992), no. 1, 61–86.
[16] D. Kazhdan and S. Patterson, *Metaplectic forms*, Inst. Hautes Études Sci. Publ. Math. No. 59 (1984), 35–142.
[17] T. Kubota, *On automorphic functions and the reciprocity law in a number field*, Kinokuniya Book-Store Co., Ltd., Tokyo 1969.
[18] S.J. Patterson, *A cubic analogue of the theta series,* J. Reine Angew. Math. 296 (1977), 125–161.
[19] S.J. Patterson, *A cubic analogue of the theta series. II,* J. Reine Angew. Math. 296 (1977), 217–220.
[20] S. Patterson, *Note on a paper of J. Hoffstein*, Glasg. Math. J. 49 (2007), no. 2, 243–255.
[21] T. Suzuki, *Rankin-Selberg convolutions of generalized theta series*, J. Reine Angew. Math. 414 (1991), 149–205.
[22] T. Suzuki, *On the biquadratic theta series*, J. Reine Angew. Math. 438 (1993), 31–85.
[23] G. Wellhausen, *Fourierkoeffizienten von Thetafunktionen sechster Ordnung*, Thesis, Georg-August-Universität zu Göttingen, 1996.

Department of Mathematics, The City College of New York, New York, NY 10031, USA
E-mail address: chinta@sci.ccny.cuny.edu