

METAPLECTIC DEMAZURE OPERATORS AND WHITTAKER FUNCTIONS

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ABSTRACT. In [CG10] the first two named authors defined an action of a Weyl group on rational functions and used it to construct multiple Dirichlet series. These series are related to Whittaker functions on an n -fold metaplectic cover of a reductive group. In this paper, we define metaplectic analogues of the Demazure and Demazure–Lusztig operators. We show how these operators can be used to recover the formulas from [CG10], and how, together with results of McNamara [McN], they can be used to compute Whittaker functions on metaplectic groups over p -adic fields.

1. INTRODUCTION

The Casselman–Shalika formula is an explicit formula for the values of the spherical Whittaker functions associated to an unramified principal series representation of a reductive group over a non-archimedian local field F [CS80], generalizing earlier work of Shintani [Shi76]. This has proven to be an important tool in the study of automorphic forms, and in particular, in the construction of L -functions. Similarly, the *metaplectic* Casselman–Shalika formula is relevant to the study of certain Dirichlet series in several complex variables that are expected to be the global Whittaker functions of Eisenstein series on metaplectic covers of reductive groups.

Three related but distinct approaches to generalizing the Casselman–Shalika formula to the nonlinear setting have recently emerged. The first is found in work of Brubaker–Bump–Friedberg [BBF11]. Working over a global field and building on earlier work with Hoffstein [BBFH07], these authors compute the Whittaker functions of the Borel Eisenstein series on a metaplectic cover of SL_r . A recursion relating Whittaker functions on a cover of SL_r to those on SL_{r-1} plays a key role in their proof. They show that though the Whittaker functions are not Euler products, they do satisfy a certain twisted multiplicativity that reduces their specification to a description of their p -parts, for p a prime. These p -parts are then shown to be expressible in terms of sums over a crystal base.

Second, the work of McNamara [McN11] treats metaplectic covers \tilde{G} of a simply-connected Chevalley group G over a local field. He directly computes the spherical Whittaker function by integrating the spherical vector φ_K over the (opposite) unipotent subgroup U^- . McNamara defines a decomposition of U^- into a collection of disjoint subsets in bijection with the (infinite) crystal graph $B(-\infty)$; on each subset the integrand φ_K is constant. This proves that the Whittaker function can be realized as a sum over a crystal base. When $G = \mathrm{SL}_r$, he recovers the formulas of Brubaker–Bump–Friedberg–Hoffstein.

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Finally, a third approach appears in the work of Chinta–Offen [CO13]. This expresses the p -adic Whittaker functions on a metaplectic cover of GL_r over a p -adic field as a sum over the Weyl group. This approach has since been generalized by McNamara [McN] to the context of tame covers of unramified reductive groups over a local field. The formulas in these works involve a “metaplectic” action of the Weyl group on rational functions. This action, which has its origins in Kazhdan–Patterson’s seminal investigation of automorphic forms on metaplectic covers of GL_r [KP84], was used by two of us (GC and PG) to construct *Weyl group multiple Dirichlet series* [CG07, CG10]. These are infinite series in several complex variables analogous to the classical Dirichlet series in one variable, such as the Riemann ζ function and Dirichlet L -functions. They satisfy a group of functional equations isomorphic to the Weyl group that intermixes the variables. A consequence of the works [CO13, McN] is that the p -adic metaplectic Whittaker functions coincide with the local factors of these series (cf. §6).

It is the formulas arising in the third approach that concern us in this article, which is partially motivated by connections between Whittaker functions and the geometry and combinatorics of Schubert varieties. In the nonmetaplectic case, that Whittaker functions on G/F are related to the geometry of the flag variety X attached to the complex dual group $\hat{G}(\mathbb{C})$ has been recently elucidated by Brubaker–Bump–Licata [BBL15], following earlier work of Reeder [Ree93]. In particular, recall that if $S \subset X$ is a Schubert variety and \mathcal{L} is a line bundle on X with global sections, then the space $H^0(S, \mathcal{L})$ is a $\hat{T}(\mathbb{C})$ -module, where $\hat{T}(\mathbb{C}) \subset \hat{G}(\mathbb{C})$ is a maximal torus. The character of such a module is called a *Demazure character*, and can be computed by applying *Demazure operators* to a highest weight monomial ([Dem74], full proof in [And85]). Then Brubaker–Bump–Licata prove (among other results) that the Iwahori Whittaker functions become Demazure characters when $q^{-1} \rightarrow 0$, where q is the cardinality of the residue field.

To generalize results of [BBL15] to the metaplectic case, a first step is developing a metaplectic analogue of the Demazure character formula. (Demazure’s version of the Weyl character formula appears in [Ful97].) This is accomplished in the present paper. We define metaplectic Demazure and *Demazure–Lusztig* operators using the metaplectic Weyl group action found in [CG10, CG07]. We prove that these operators satisfy the same relations as their classical counterparts. We also prove an analogue of Demazure’s version of the Weyl character formula (corresponding to the case $S = X$ above) (Theorem 3), as well as a companion identity for the Demazure–Lusztig operators (Theorem 4) and show how they can be used to compute spherical Whittaker functions on metaplectic covers (Theorem 16).

We anticipate several directions of future research based on the present work, which we now describe.

Combinatorial applications. The results of this article provide a combinatorial tool compatible with the work of Brubaker–Bump–Friedberg, McNamara and Chinta–Offen described above. This suggests their utility in unifying these distinct approaches to Whittaker functions. The third named author applies Theorem 3 and Theorem 4 to establish a combinatorial bridge between the formulas resulting from [BBF11, McN11] and [CO13, McN] in Dynkin type A [Pus14]. When the cover is trivial, these formulas are related by a theorem of Tokuyama [Tok88]. A generalization of Tokuyama’s theorem in terms of metaplectic Demazure–Lusztig operators provides the desired link.

An other possible application of our combinatorial identities is extending the approach of [BBF11, McN11] to other Dynkin types. Various authors have explored this direction

(Beineke [BBF12], Brubaker, Bump, Chinta, Gunnells [BBCG12], Frechette, Friedberg and Zhang [FZ13] for type B and C , Chinta and Gunnells for type D [CG12]). These sources describe individual coefficients of a Whittaker function explicitly. However, the resulting expressions are more intricate than in type A . One hopes that interpreting the constructions in terms of metaplectic Demazure–Lusztig operators allows for a more transparent and uniform combinatorial description in all types.

Double covers. In the case of the double cover, metaplectic Whittaker functions often parametrize objects related to quadratic rings or representation numbers of quadratic forms. For example,

- the global Whittaker functions on a double cover of $SL_2(\mathbb{R})$ can be expressed in terms of Dedekind zeta functions of quadratic number fields. Consequently, they count ideals in quadratic fields.
- For $SL_3(\mathbb{R})$, the global Whittaker functions form a double Dirichlet series which counts the number of ideals of norm n in the quadratic extension of discriminant d .
- As shown by Wen [Wen], the coefficients of global Whittaker functions on the double cover of $SL_4(\mathbb{R})$ are related to counting $2 \times 2 \times 2$ integer cubes modulo the action of a certain subgroup of $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$. By the work of Bhargava [Bha04], these cubes are also related to the arithmetic of ideals in quadratic number fields.

In the latter two cases, the global Whittaker functions are not Euler products. Nevertheless they are built out of the local Whittaker functions using twisted multiplicativity, as described in [BBC⁺06]. The Demazure operators introduced in the present paper act on the coefficients of these local generating series. It is natural to ask whether the operators act in a natural way on the arithmetic objects enumerated by the coefficients. We hope to return to this topic in a future work.

Iwahori–Whittaker functions. Brubaker–Bump–Licata [BBL15] express nonmetaplectic Iwahori–Whittaker functions recursively in terms of variants of Demazure–Lusztig operators. In recent work, Patnaik and Puskás use the tools introduced in this article, and earlier work by Patnaik [Pat] to express metaplectic Iwahori–Whittaker functions in terms of Demazure–Lusztig operators. The presence of a direct connection between our operators and Iwahori–Whittaker functions indicates that the same methods might also prove useful in the study of Whittaker functions in other settings, such as affine groups.

Affine groups. As we have already indicated there are now several different approaches to the study of metaplectic Whittaker functions (McNamara [McN11, McN], Chinta–Offen [CO13], Brubaker–Bump–Friedberg–Hoffstein [BBF11, BBFH07]). A important direction for future research is the extension of this theory to affine groups. Several recent results in the theory of multiple Dirichlet series (see e.g. Bucur–Diaconu [BD10], Lee–Zhang [LZ15], Whitehead [Whi14]) hint at a theory of metaplectic Whittaker function on affine groups rich in applications. Recent work of Patnaik develops a Casselman–Shalika formula for Whittaker functions on affine groups [Pat]. His proof makes use of Demazure–Lusztig operators. We are hopeful that the ideas introduced in our paper can similarly be extended to study metaplectic Whittaker functions on affine groups. This topic is currently being investigated by Patnaik and Puskás.

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2. NOTATION

We begin by setting up notation. For unexplained notions about root systems and Coxeter groups, we refer to [Bou02].

Let Φ be an irreducible reduced root system of rank r with Weyl group W . Choose an ordering of the roots and let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into positive and negative roots. Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be the set of simple roots, and let σ_i be the Weyl group element corresponding to the reflection through the hyperplane perpendicular to α_i . Define

$$(1) \quad \Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}.$$

Let Λ be a lattice containing Φ as a subset. Later (Section 6) we will assume that Λ is the coweight lattice of a split reductive algebraic group G defined over the non-archimedean field F , and Φ is its coroots, but at the moment this is not necessary. Right now all we require is that the Weyl group W acts on Λ , and that there is a W -invariant \mathbb{Z} -valued quadratic form Q defined on Λ . Define a bilinear form $B(\alpha, \beta)$ by $Q(\alpha + \beta) - Q(\alpha) - Q(\beta)$.

We fix a positive integer n . The integer n determines a collection of integers $\{m(\alpha) : \alpha \in \Phi\}$ by

$$(2) \quad m(\alpha) = n / \gcd(n, Q(\alpha)),$$

and a sublattice $\Lambda_0 \subset \Lambda$ by

$$(3) \quad \Lambda_0 = \{\lambda \in \Lambda : B(\alpha, \lambda) \equiv 0 \pmod{n} \text{ for all simple roots } \alpha\}.$$

With these definitions, one can easily prove the following:

Lemma 1. *For any simple root α , we have $m(\alpha)\alpha \in \Lambda_0$.* □

Let $\mathcal{A} = \mathbb{C}[\Lambda]$ be the ring of Laurent polynomials on Λ and \mathcal{K} its field of fractions. The action of W on the lattice Λ induces an action of W on \mathcal{K} : we put

$$(4) \quad (w, x^\lambda) \longmapsto x^{w\lambda} =: w.x^\lambda,$$

and then extend linearly and multiplicatively to all of \mathcal{K} . We will always denote this action using the lower dot

$$(w, f) \longmapsto w.f$$

to distinguish it from the metaplectic W -action on \mathcal{K} constructed below in (9).

Let $\lambda \mapsto \bar{\lambda}$ be the projection $\Lambda \rightarrow \Lambda/\Lambda_0$ and $(\Lambda/\Lambda_0)^*$ be the group of characters of the quotient lattice. Any $\xi \in (\Lambda/\Lambda_0)^*$ induces a field isomorphism of \mathcal{K}/\mathbb{C} by setting $\xi(x^\lambda) = \xi(\bar{\lambda}) \cdot x^\lambda$ for $\lambda \in \Lambda$. This leads to the direct sum decomposition

$$(5) \quad \mathcal{K} = \bigoplus_{\bar{\lambda} \in \Lambda/\Lambda_0} \mathcal{K}_{\bar{\lambda}}$$

where $\mathcal{K}_{\bar{\lambda}} = \{f \in \mathcal{K} : \xi(f) = \xi(\bar{\lambda}) \cdot f \text{ for all } \xi \in (\Lambda/\Lambda_0)^*\}$

Next choose nonzero complex parameters v, g_0, \dots, g_{n-1} satisfying

$$(6) \quad g_0 = -1 \text{ and } g_i g_{n-i} = v^{-1} \text{ for } i = 1, \dots, n-1;$$

for all other j we define $g_j := g_{r_n(j)}$, where $0 \leq r_n(j) < n-1$ denotes the remainder upon dividing j by n . Introduce the following deformation of the Weyl denominator:

$$\Delta_v = \prod_{\alpha \in \Phi^+} (1 - v \cdot x^{m(\alpha)\alpha}).$$

If $v = 1$ we write more simply $\Delta_v = \Delta$.

We now define an action of the Weyl group W on \mathcal{K} as follows. For $f \in \mathcal{K}_{\bar{\lambda}}$ and $\sigma_\alpha \in W$ the generator corresponding to a simple root α , define

$$(7) \quad \sigma_i(f) = \frac{\sigma_i \cdot f}{1 - vx^{m(\alpha_i)\alpha_i}} \cdot \left[x^{-r_{m(\alpha_i)}\left(-\frac{B(\lambda, \alpha_i)}{Q(\alpha_i)}\right) \cdot \alpha_i} \cdot (1 - v) \right. \\ \left. - v \cdot g_{Q(\alpha_i) - B(\lambda, \alpha_i)} \cdot x^{(1-m(\alpha_i))\alpha_i} \cdot (1 - x^{m(\alpha_i)\alpha_i}) \right]$$

where λ is any lift of $\bar{\lambda}$ to Λ . It is easy to see that the quantity in brackets depends only on $\bar{\lambda}$. We extend the definition of σ_α to \mathcal{K} by additivity. One can check that with this definition, $\sigma_\alpha^2(f) = f$ for all $f \in \mathcal{K}$. Furthermore it is proven in [CG10] (see also [McN]) that this action satisfies the defining relations of W : if $(m_{i,j})$ is the Coxeter matrix for Φ , then

$$(8) \quad (\sigma_i \sigma_j)^{m_{i,j}}(f) = f \quad \text{for all } i, j \text{ and } f \in \mathcal{K}.$$

Therefore (7) extends to an action of the full Weyl group W on \mathcal{K} , which we denote

$$(9) \quad (w, f) \longmapsto w(f).$$

We remark that if $n = 1$, the action (7) collapses to the usual action (4) of W on \mathcal{K} . That the quantity in brackets in (7) depends only on $\bar{\lambda}$ and not λ translates to the following lemma:

Lemma 2. *Let $f \in \mathcal{K}$ and $h \in \mathcal{K}_0$. Then for any $w \in W$,*

$$w(hf) = (w.h) \cdot w(f).$$

Here $w.h$ means the action of (4), whereas \cdot denotes multiplication in \mathcal{K} . □

Lemma 2 is used repeatedly in the proofs below. It is important to note that the action of W on \mathcal{K} defined by (7) is \mathbb{C} -linear, but is *not* by endomorphisms of that ring, i.e. it is not in general multiplicative. The point of Lemma 2 is that if we have a product of two terms hf , the first of which satisfies $h \in \mathcal{K}_0$, then in (7) we can apply w to the product hf by performing the usual permutation action on h and then acting on f by the twisted W -action.

Next we use this Weyl group action to define certain divided difference operators. For $1 \leq i \leq r$ and $f \in \mathcal{K}$ define the *Demazure operators* by

$$(10) \quad \mathcal{D}_i(f) = \mathcal{D}_{\sigma_i}(f) = \frac{f - x^{m(\alpha_i)\alpha_i} \cdot \sigma_i(f)}{1 - x^{m(\alpha_i)\alpha_i}},$$

and the *Demazure–Lusztig operators*¹ by

$$(11) \quad \begin{aligned} \mathcal{T}_i(f) &= \mathcal{T}_{\sigma_i}(f) = \left(1 - v \cdot x^{m(\alpha_i)\alpha_i}\right) \cdot \mathcal{D}_i(f) - f \\ &= \left(1 - v \cdot x^{m(\alpha_i)\alpha_i}\right) \cdot \frac{f - x^{m(\alpha_i)\alpha_i} \cdot \sigma_i(f)}{1 - x^{m(\alpha_i)\alpha_i}} - f. \end{aligned}$$

When there is no danger of confusion, we write more simply

$$\mathcal{D}_i = \frac{1 - x^{m(\alpha_i)\alpha_i} \sigma_i}{1 - x^{m(\alpha_i)\alpha_i}} \text{ and } \mathcal{T}_i = \left(1 - v \cdot x^{m(\alpha_i)\alpha_i}\right) \cdot \mathcal{D}_i - 1,$$

¹We remark that our notation is somewhat nonstandard, since in the usual normalization averaging the Demazure–Lusztig operators over the Weyl group produces Macdonald’s formula for the spherical function, not the Casselman–Shalika formula. We thank the referee for pointing this out.

that is, a rational function h in the above equations is interpreted to mean the “multiplication by h ” operator. The rational functions here are in \mathcal{K}_0 .

We prove in the following section that the operators \mathcal{D}_i and \mathcal{T}_i satisfy the same braid relations as the σ_i . Consequently, we can define \mathcal{D}_w and \mathcal{T}_w for any $w \in W$ as follows. Let $w = \sigma_{i_1} \cdots \sigma_{i_l}$ be a reduced expression for w in terms of simple reflections. Then we define

$$\mathcal{D}_w = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_l} \quad \text{and} \quad \mathcal{T}_w = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_l}.$$

In the first two theorems below, both sides of the equalities are to be understood as identities of operators on \mathcal{K} .

Theorem 3. *For the long element w_0 of the Weyl group W we have*

$$\mathcal{D}_{w_0} = \frac{1}{\Delta} \cdot \sum_{w \in W} \text{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha} \cdot w.$$

Theorem 4. *We have*

$$\Delta_v \cdot \mathcal{D}_{w_0} = \sum_{w \in W} \mathcal{T}_w.$$

We prove Theorem 3 in Section 4 and Theorem 4 in Section 5.

Remark 1. In Section 6 we use the work of McNamara [McN] to express Whittaker functions over a local p -adic field in terms of the operators introduced above. In this section the parameters will be specialized: v will be set to equal q^{-1} (for q the cardinality of the residue field) and the g_i will be Gauss sums. For now, we only need these parameters to satisfy the relations (6).

3. BASIC PROPERTIES OF THE OPERATORS

In this section we prove the *quadratic relations* (Proposition 5) and *braid relations* (Proposition 7) satisfied by the Demazure and Demazure–Lusztig operators.

Proposition 5. *The operators \mathcal{D}_i and \mathcal{T}_i ($1 \leq i \leq r$) satisfy the following quadratic relations:*

- (i) $\mathcal{D}_i^2 = \mathcal{D}_i$;
- (ii) $\mathcal{T}_i^2 = (v - 1)\mathcal{T}_i + v$.

Proof. We prove (i) in detail and leave (ii) to the reader. To simplify the notation, we drop the subscripts and write \mathcal{D} , α , and σ , and abbreviate $m(\alpha)$ to m . Using the definition of \mathcal{D} and Lemma 2, we have

$$\begin{aligned} \mathcal{D}^2 &= \left(\frac{1 - x^{m\alpha}\sigma}{1 - x^{m\alpha}} \right)^2 \\ &= \left(\frac{1}{(1 - x^{m\alpha})^2} + \frac{x^{m\alpha}}{1 - x^{m\alpha}} \cdot \frac{x^{-m\alpha}}{1 - x^{-m\alpha}} \right) \cdot 1 \\ &\quad + \left(\frac{-x^{m\alpha}}{(1 - x^{m\alpha})^2} + \frac{-x^{m\alpha}}{1 - x^{m\alpha}} \cdot \frac{1}{1 - x^{-m\alpha}} \right) \cdot \sigma \\ &= \frac{1}{1 - x^{m\alpha}} \cdot \left(\frac{1}{1 - x^{m\alpha}} + \frac{1}{1 - x^{-m\alpha}} \right) \cdot 1 \\ &\quad + \frac{-x^{m\alpha}}{1 - x^{m\alpha}} \cdot \left(\frac{1}{1 - x^{m\alpha}} + \frac{1}{1 - x^{-m\alpha}} \right) \cdot \sigma. \end{aligned}$$

Since

$$\frac{1}{1-x^{m\alpha}} + \frac{1}{1-x^{-m\alpha}} = 1$$

we obtain $\mathcal{D}^2 = \mathcal{D}$. □

We pause to point out the key role played by Lemma 2 in the proof of Proposition 5: the action of σ_i on an arbitrary rational function is given by the complicated formula (7), but thanks to Lemma 2 we can pass the operator σ past the monomial $x^{m(\alpha)\alpha}$, after acting on this monomial by the usual permutation action. This fact will be used repeatedly throughout the paper.

Lemma 6. *We have $\mathcal{D}_i x^{m(\alpha_i)\alpha_i} \mathcal{D}_i = -\mathcal{D}_i$.*

Proof. We use the same notation as in the proof of Proposition 5 and compute directly:

$$\begin{aligned} \mathcal{D} x^{m\alpha} \mathcal{D} &= \mathcal{D} \left(\frac{x^{m\alpha} - x^{2m\alpha}\sigma}{1 - x^{m\alpha}} \right) \\ &= \frac{x^{m\alpha} - x^{2m\alpha}\sigma}{(1 - x^{m\alpha})^2} - \frac{\sigma - x^{-m\alpha}}{(1 - x^{m\alpha})(1 - x^{-m\alpha})} \\ &= \frac{x^{m\alpha}\sigma - 1}{1 - x^{m\alpha}} \\ &= -\mathcal{D}. \end{aligned}$$

□

Proposition 7. *Suppose $(\sigma_i\sigma_j)^{m_{i,j}} = 1$ is a defining relation for W . Then*

$$(12) \quad \mathcal{D}_i \mathcal{D}_j \mathcal{D}_i \cdots = \mathcal{D}_j \mathcal{D}_i \mathcal{D}_j \cdots,$$

$$(13) \quad \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \cdots = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \cdots,$$

where there are $m_{i,j}$ factors on both sides of (12)–(13).

Proof. Both statements boil down to explicit computations with rank 2 root systems, and in fact are special cases of Theorems 3 and 4. We explain what happens in detail with (12) in A_2 , which is typical of all the computations. Since all roots have the same length, we lighten notation by putting $m = m(\alpha)$.

By definition,

$$\mathcal{D}_1 = \frac{1 - x^{m\alpha_1}\sigma_1}{1 - x^{m\alpha_1}}.$$

Next we apply \mathcal{D}_2 and use $\sigma_2(\alpha_1) = \alpha_1 + \alpha_2$:

$$\mathcal{D}_2 \mathcal{D}_1 = \frac{1 - x^{m\alpha_1}\sigma_1}{(1 - x^{m\alpha_2})(1 - x^{m\alpha_1})} - \frac{x^{m\alpha_2}\sigma_2 - x^{m(\alpha_1+2\alpha_2)}\sigma_2\sigma_1}{(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})}$$

Finally we apply \mathcal{D}_1 to obtain

$$\begin{aligned} \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_1 &= \frac{1 - x^{m\alpha_1}\sigma_1}{(1 - x^{m\alpha_2})(1 - x^{m\alpha_1})^2} - \frac{x^{m\alpha_2}\sigma_2 - x^{m(\alpha_1+2\alpha_2)}\sigma_2\sigma_1}{(1 - x^{m\alpha_1})(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})} \\ &\quad - \left(\frac{x^{m\alpha_1}\sigma_1 - 1}{(1 - x^{m\alpha_1})(1 - x^{m(\alpha_1+\alpha_2)})(1 - x^{-m\alpha_1})} - \frac{x^{m(2\alpha_1+\alpha_2)}\sigma_1\sigma_2 - x^{m(2\alpha_1+2\alpha_2)}\sigma_1\sigma_2\sigma_1}{(1 - x^{m\alpha_1})(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1+\alpha_2)})} \right), \end{aligned}$$

which simplifies to

$$(14) \quad \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_1 = \frac{1 - x^{m\alpha_1} \sigma_1 - x^{m\alpha_2} \sigma_2 + x^{m(2\alpha_1 + \alpha_2)} \sigma_1 \sigma_2 + x^{m(\alpha_1 + 2\alpha_2)} \sigma_2 \sigma_1 - x^{m(2\alpha_1 + 2\alpha_2)} \sigma_1 \sigma_2 \sigma_1}{\Delta},$$

where $\Delta = (1 - x^{m\alpha_1})(1 - x^{m\alpha_2})(1 - x^{m(\alpha_1 + \alpha_2)})$. The final formula (14) clearly depends only on the longest word in the Weyl group for A_2 and not on the reduced expression used to define it, which proves (12). (Note that this computation also checks Theorem 3 for $\Phi = A_2$.) \square

4. PROOF OF THEOREM 3

We now turn to the proof of Theorem 3. Before we can begin, we require more notation. The following is [Bum04, Proposition 21.10], applied to $\Phi(w^{-1})$ instead of $\Phi(w)$:

Proposition 8. *Let $w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N}$ be a reduced expression for $w \in W$. Then the set*

$$\Phi(w^{-1}) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^-\}$$

consists of the elements

$$\alpha_{i_1}, \sigma_{i_1}(\alpha_{i_2}), \sigma_{i_1} \sigma_{i_2}(\alpha_{i_3}), \dots, \sigma_{i_1} \cdots \sigma_{i_{N-1}}(\alpha_{i_N}),$$

where the α_i are the simple roots.

Let $p: \Phi \rightarrow \mathcal{K}_0$ be a map. We say p is W -intertwining if for any $\beta \in \Phi$ and $w \in W$, we have

$$p(w\beta) = w.p(\beta).$$

Proposition 8 has the following corollary, useful for the proof of Theorems 3 and 4:

Corollary 9. *Assume $p: \Phi \rightarrow \mathcal{K}_0$ is W -intertwining, and suppose $w \in W$ has a reduced expression $w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N}$. Then we have the following equality of operators on \mathcal{K} :*

$$(15) \quad p(\alpha_{i_1}) \sigma_{i_1} \cdot p(\alpha_{i_2}) \sigma_{i_2} \cdots p(\alpha_{i_N}) \sigma_{i_N} = \left(\prod_{\alpha \in \Phi(w^{-1})} p(\alpha) \right) \cdot w.$$

Proof. Making repeated use of Lemma 2, we can re-order the operators on the left of (15) by passing all the σ_{i_j} s to the right and all elements of \mathcal{K}_0 to the left. After this, the left of (15) becomes

$$p(\alpha_{i_1}) \cdot (\sigma_{i_1} \cdot p(\alpha_{i_2})) \cdot (\sigma_{i_1} \sigma_{i_2} \cdot p(\alpha_{i_3})) \cdots (\sigma_{i_1} \cdots \sigma_{i_{N-1}} \cdot p(\alpha_{i_N})) \cdot \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N}.$$

Here $\sigma_{i_1} \cdots \sigma_{i_N}$ is a reduced expression for w . Moreover, by Proposition 8,

$$\alpha_{i_1}, \sigma_{i_1}(\alpha_{i_2}), \sigma_{i_1} \sigma_{i_2}(\alpha_{i_3}), \dots, \sigma_{i_1} \cdots \sigma_{i_{N-1}}(\alpha_{i_N})$$

enumerates $\Phi(w^{-1})$. As a consequence the corresponding elements of \mathcal{K}_0 , namely

$$p(\alpha_{i_1}), p(\sigma_{i_1}(\alpha_{i_2})), p(\sigma_{i_1} \sigma_{i_2}(\alpha_{i_3})), \dots, p(\sigma_{i_1} \cdots \sigma_{i_{N-1}}(\alpha_{i_N})),$$

have product $\prod_{\alpha \in \Phi(w^{-1})} p(\alpha)$. Since the map p is W -intertwining, these are exactly the factors appearing on the left of (15). \square

We now begin the proof of Theorem 3. First notice that by Lemma 2, any composition of the operators \mathcal{D}_i can be written as a \mathcal{K}_0 -linear combination of the operators $w \in W$. Hence we can write

$$(16) \quad \mathcal{D}_{w_0} = \sum_{w \in W} R_w \cdot w,$$

for some choice of rational functions $R_w \in \mathcal{K}_0$.

Let $l: W \rightarrow \mathbb{Z}$ denote the length function on W . It is a standard fact about finite Coxeter groups that for any $1 \leq j \leq r$, we can find a reduced expression $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_{l(w_0)}}$ for the longest word w_0 with $i_1 = j$. By Proposition 7, we have $\mathcal{D}_{w_0} = \mathcal{D}_j \mathcal{D}_{w_j}$ for $w_j = \sigma_{i_2} \dots \sigma_{i_{l(w_0)}}$. Since $\mathcal{D}_j^2 = \mathcal{D}_j$ (Proposition 5), we have $\mathcal{D}_j \mathcal{D}_{w_0} = \mathcal{D}_{w_0}$. In other words,

$$\frac{1 - x^{m(\alpha_j)\alpha_j} \sigma_j}{1 - x^{m(\alpha_j)\alpha_j}} \mathcal{D}_{w_0} = \mathcal{D}_{w_0}.$$

It follows that $\sigma_j \mathcal{D}_{w_0} = \mathcal{D}_{w_0}$. Now apply σ_j to both sides of (16). Since each $R_w \in \mathcal{K}_0$, Lemma 2 implies

$$(17) \quad \mathcal{D}_{w_0} = \sum_{w \in W} (\sigma_j \cdot R_w) \cdot \sigma_j w.$$

Comparing coefficients in (16) and (17) and using the fact that the elements of W are linearly independent as operators on \mathcal{K} , we obtain $\sigma_j \cdot R_w = R_{\sigma_j w}$. Thus

$$(18) \quad u \cdot R_w = R_{uw} \quad \forall u, w \in W$$

To finish the proof of Theorem 3, it suffices to compute R_{w_0} ; the remaining coefficients can then be computed using (18). In fact we shall prove the following:

Lemma 10. *For $w \in W$, we have*

$$R_w = \frac{\text{sgn}(w)}{\Delta} \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}.$$

Proof. To start, assume the statement is true for $w = w_0$:

$$(19) \quad R_{w_0} = \frac{\text{sgn}(w_0)}{\Delta} \prod_{\alpha \in \Phi^+} x^{m(\alpha)\alpha}.$$

By (18), we have

$$\begin{aligned} R_{uw_0} &= u \cdot R_{w_0} = u \left(\text{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} \frac{x^{m(\alpha)\alpha}}{1 - x^{m(\alpha)\alpha}} \right) \\ &= \text{sgn}(w_0) \cdot \prod_{\alpha \in u(\Phi^+)} \frac{x^{m(\alpha)\alpha}}{1 - x^{m(\alpha)\alpha}} \\ &= R_{w_0} \text{sgn}(u) \prod_{\alpha \in \Phi(u^{-1})} x^{-m(\alpha)\alpha} \end{aligned}$$

Let $u = ww_0$. Then $\Phi(u^{-1}) = \Phi^+ \cap w(\Phi^+)$. Hence

$$\begin{aligned} R_w &= R_{w_0} \operatorname{sgn}(ww_0) \prod_{\alpha \in \Phi^+ \cap w(\Phi^+)} x^{-m(\alpha)\alpha} \\ &= \frac{\operatorname{sgn}(w)}{\Delta} \prod_{\alpha \in \Phi^+} x^{m(\alpha)\alpha} \prod_{\alpha \in \Phi^+ \cap w(\Phi^+)} x^{-m(\alpha)\alpha} \\ &= \frac{\operatorname{sgn}(w)}{\Delta} \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}. \end{aligned}$$

Thus the proof will be complete if we show (19).

Begin by writing

$$\mathcal{D}_i = p_1(\alpha_i) - p_2(\alpha_i)\sigma_i,$$

where $p_1, p_2: \Phi \rightarrow \mathcal{K}_0$ are defined by

$$p_1(\beta) = \frac{1}{1 - x^{m(\beta)\beta}}, \quad p_2(\beta) = \frac{x^{m(\beta)\beta}}{1 - x^{m(\beta)\beta}}.$$

Given a reduced expression $w_0 = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_N}$, it is easy to see that

$$R_{w_0}w_0 = \operatorname{sgn}(w_0)p_2(\alpha_{i_1}) \cdot \sigma_{i_1} \cdot p_2(\alpha_{i_2}) \cdot \sigma_{i_2} \cdots p_2(\alpha_{i_N}) \cdot \sigma_{i_N}.$$

Since the map p_2 is readily shown to be W -intertwining, it follows from the above equality and Corollary 9 that

$$R_{w_0} = \operatorname{sgn}(w) \prod_{\alpha \in \Phi^+} p_2(\alpha) = \frac{\operatorname{sgn}(w_0)}{\Delta} \prod_{\alpha \in \Phi^+} x^{m(\alpha)\alpha}.$$

This completes the proof of the lemma, and thus of Theorem 3. \square

5. PROOF OF THEOREM 4

In this section we prove Theorem 4:

$$\Delta_v \cdot \mathcal{D}_{w_0} = \sum_{w \in W} \mathcal{T}_w.$$

Let $\tilde{\mathcal{T}} = \sum_{w \in W} \mathcal{T}_w$. We begin with some lemmas.

Lemma 11. *For any $1 \leq i \leq r$ we have*

$$\mathcal{T}_i \cdot (\Delta_v \mathcal{D}_{w_0}) = v \cdot (\Delta_v \mathcal{D}_{w_0})$$

Proof. Since the simple reflection σ_i permutes the elements of $\Phi^+ \setminus \{\alpha_i\}$, the operator \mathcal{D}_i commutes with

$$\prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} (1 - vx^{m(\beta)\beta}) = \frac{\Delta_v}{1 - vx^{m(\alpha_i)\alpha_i}}.$$

Consequently,

$$(20) \quad (1 - vx^{m(\alpha_i)\alpha_i}) \mathcal{D}_i \Delta_v = \Delta_v \mathcal{D}_i (1 - vx^{m(\alpha_i)\alpha_i}).$$

Take a reduced expression for the long element, $w_0 = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_N}$ satisfying $i_1 = i$. Thus $\mathcal{D}_{w_0} = \mathcal{D}_i \mathcal{D}_{w_i}$ for $w_i = \sigma_{i_2} \cdots \sigma_{i_N}$. Using this and (20),

$$(\mathcal{T}_i + 1) \cdot (\Delta_v \mathcal{D}_{w_0}) = \Delta_v \mathcal{D}_i (1 - vx^{m(\alpha_i)\alpha_i}) \mathcal{D}_i \mathcal{D}_{w_i}.$$

The idempotency of \mathcal{D}_i (Proposition 5) and Lemma 6 imply

$$\mathcal{D}_i(1 - vx^{m(\alpha_i)\alpha_i})\mathcal{D}_i = (1 + v)\mathcal{D}_i.$$

Putting everything together, we conclude that $(\mathcal{T}_i + 1) \cdot (\Delta_v \mathcal{D}_{w_0}) = (1 + v) \cdot (\Delta_v \mathcal{D}_{w_0})$. \square

Lemma 12. *For any $1 \leq i \leq r$ we have*

$$\mathcal{T}_i \cdot \tilde{\mathcal{T}} = v \cdot \tilde{\mathcal{T}}.$$

Proof. Recall that $l: W \rightarrow \mathbb{Z}$ is the length function on W . Then for any element $w \in W$ and any simple reflection σ_i , we have $l(\sigma_i w) = l(w) \pm 1$. Partition W into $C_1 \cup C_2$, where

$$\begin{aligned} C_1 &= \{w \in W : l(\sigma_i w) = l(w) - 1\}, \\ C_2 &= \{w \in W : l(\sigma_i w) = l(w) + 1\}. \end{aligned}$$

Then the map $w \mapsto \sigma_i w$ defines a bijection between C_1 and C_2 . We compute

$$\begin{aligned} \mathcal{T}_i \tilde{\mathcal{T}} &= \mathcal{T}_i \cdot \left(\sum_{w \in C_1} \mathcal{T}_w + \sum_{w \in C_2} \mathcal{T}_w \right) \\ &= \mathcal{T}_i \cdot \left(\sum_{w \in C_2} \mathcal{T}_i \mathcal{T}_w + \sum_{w \in C_2} \mathcal{T}_w \right) \\ &= \sum_{w \in C_2} \mathcal{T}_i^2 \mathcal{T}_w + \sum_{w \in C_2} \mathcal{T}_i \mathcal{T}_w. \end{aligned}$$

The second sum above is simply $\sum_{w \in C_1} \mathcal{T}_w$. In the first, we use the quadratic relation $\mathcal{T}_i^2 = (v - 1)\mathcal{T}_i + v$ of Proposition 5 to write

$$\sum_{w \in C_2} \mathcal{T}_i^2 \mathcal{T}_w = (v - 1) \sum_{w \in C_2} \mathcal{T}_i \mathcal{T}_w + v \sum_{w \in C_2} \mathcal{T}_w = (v - 1) \sum_{w \in C_1} \mathcal{T}_w + v \sum_{w \in C_2} \mathcal{T}_w.$$

Thus $\mathcal{T}_i \cdot \tilde{\mathcal{T}} = v \cdot \tilde{\mathcal{T}}$. \square

Lemma 13. *Let*

$$\tilde{\mathcal{R}} = \sum_{w \in W} R_w \cdot w$$

be an operator on \mathcal{K} that is a linear combination of the Weyl group elements with coefficients $R_w \in \mathcal{K}_0$. Assume $\tilde{\mathcal{R}}$ is an eigenclass for \mathcal{T}_i with eigenvalue v :

$$\mathcal{T}_i \cdot \tilde{\mathcal{R}} = v \cdot \tilde{\mathcal{R}}.$$

Then for every $w \in W$, we have

$$R_w = \frac{1 - vx^{m(\alpha_i)\alpha_i}}{1 - vx^{-m(\alpha_i)\alpha_i}} \cdot \sigma_i \cdot R_{\sigma_i w}.$$

Proof. The proof is a straightforward computation. Begin with

$$\mathcal{T}_i R_w w = [q_1(\alpha_i) - q_2(\alpha_i)\sigma_i] R_w w = q_1(\alpha_i) R_w w - q_2(\alpha_i)(\sigma_i \cdot R_w) \sigma_i w$$

with q_1, q_2 defined by

$$(21) \quad q_1(\beta) = \frac{1 - vx^{m(\beta)\beta}}{1 - x^{m(\beta)\beta}} - 1, \quad \text{and} \quad q_2(\beta) = \frac{(1 - vx^{m(\beta)\beta})x^{m(\beta)\beta}}{1 - x^{m(\beta)\beta}}.$$

Summing over $w \in W$, we get

$$\mathcal{T}_i \cdot \tilde{\mathcal{R}} = \sum_{w \in W} [q_1(\alpha_i)R_w - q_2(\alpha_i)\sigma_i \cdot R_{\sigma_i w}] w$$

But we also have $\mathcal{T}_i \cdot \tilde{\mathcal{R}} = v \cdot \tilde{\mathcal{R}}$, so comparing coefficients yields

$$q_1(\alpha_i)R_w - q_2(\alpha_i)\sigma_i \cdot R_{\sigma_i w} = vR_w.$$

Solving for R_w completes the proof. \square

Lemma 13 has the following easy and useful corollary.

Corollary 14. *Let*

$$\tilde{\mathcal{R}} = \sum_{w \in W} R_w \cdot w, \quad \tilde{\mathcal{S}} = \sum_{w \in W} S_w \cdot w$$

be two operators on \mathcal{K} that are linear combinations of the Weyl-group elements with coefficients $R_w, S_w \in \mathcal{K}_0$. Assume that

$$\mathcal{T}_i \cdot \tilde{\mathcal{R}} = v \cdot \tilde{\mathcal{R}}, \quad \mathcal{T}_i \cdot \tilde{\mathcal{S}} = v \cdot \tilde{\mathcal{S}}$$

for every $i, 1 \leq i \leq r$. Assume further that we have $R_{w_0} = S_{w_0}$ for the long element $w_0 \in W$. Then $\tilde{\mathcal{R}} = \tilde{\mathcal{S}}$ as operators on $\tilde{\mathcal{A}}$.

Proof. We show that $R_w = S_w$ for every $w \in W$. This can be seen by descending induction on the length of w . For $l(w)$ maximal we have $R_{w_0} = S_{w_0}$ by assumption. Now assume $l(\sigma_i w) = l(w) + 1$, and $R_{\sigma_i w} = S_{\sigma_i w}$. It follows from Lemma 13 that

$$R_w = \frac{1 - vx^{m(\alpha_i)\alpha_i}}{1 - vx^{-m(\alpha_i)\alpha_i}} \cdot \sigma_i \cdot R_{\sigma_i w} = \frac{1 - vx^{m(\alpha_i)\alpha_i}}{1 - vx^{-m(\alpha_i)\alpha_i}} \cdot \sigma_i \cdot S_{\sigma_i w} = S_w,$$

thus $R_w = S_w$. This completes the proof. \square

We now turn to the proof of Theorem 4. Applying Lemmas 11 and 12 to the operators $\Delta_v \mathcal{D}_{w_0}$ and $\tilde{\mathcal{T}}$, we have

$$\mathcal{T}_i \cdot (\Delta_v \mathcal{D}_{w_0}) = v \cdot \Delta_v \mathcal{D}_{w_0}, \quad \mathcal{T}_i \cdot \tilde{\mathcal{T}} = v \cdot \tilde{\mathcal{T}}$$

for every $1 \leq i \leq r$. It follows from the definitions that as operators on \mathcal{K} , both $\Delta_v \mathcal{D}_{w_0}$ and $\tilde{\mathcal{T}}$ can be written as a linear combination of elements of W with coefficients in \mathcal{K}_0 . Let us write

$$\Delta_v \mathcal{D}_{w_0} = \sum_{w \in W} R_w \cdot w \quad \text{and} \quad \tilde{\mathcal{T}} = \sum_{w \in W} S_w \cdot w$$

for some $R_w, S_w \in \mathcal{K}$. We shall show that if $w_0 \in W$ is the long element of the Weyl group, then $R_{w_0} = S_{w_0}$. By Corollary 14, this suffices to prove the theorem.

The long coefficient R_{w_0} of $\Delta_v \mathcal{D}_{w_0}$ is easily read off from Theorem 3:

$$(22) \quad R_{w_0} = \text{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} \frac{(1 - v \cdot x^{m(\alpha)\alpha}) \cdot x^{m(\alpha)\alpha}}{(1 - x^{m(\alpha)\alpha})}.$$

To determine the coefficient S_{w_0} we again use the property of W -intertwining maps from Corollary 9 and argue as in the proof of Lemma 10. First, note that the only term in $\tilde{\mathcal{T}} = \sum_{w \in W} T_w = \sum_{w \in W} S_w \cdot w$ that contributes to the coefficient S_{w_0} is T_{w_0} . (All the

other T_w have fewer than $l(w_0)$ simple reflections appearing in them.) To examine T_{w_0} , fix a reduced expression for the long element: $w_0 = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N}$. Let us again write

$$\mathcal{T}_i = q_1(\alpha_i) - q_2(\alpha_i)\sigma_i$$

where $q_1, q_2: \Phi \rightarrow \mathcal{K}_0$ are defined in (21). It is clear that the map q_2 is W -intertwining. The only contribution to S_{w_0} from $T_{w_0} = \mathcal{T}_{i_1} \mathcal{T}_{i_2} \cdots \mathcal{T}_{i_N}$ is from

$$q_2(\alpha_{i_1})\sigma_{i_1} \cdot q_2(\alpha_{i_2})\sigma_{i_2} \cdots q_2(\alpha_{i_N})\sigma_{i_N}.$$

Using Corollary 9, we conclude

$$(23) \quad S_{w_0} = \text{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} q_2(\alpha) = \text{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} \frac{(1 - v \cdot x^{m(\alpha)\alpha}) \cdot x^{m(\alpha)\alpha}}{1 - x^{m(\alpha)\alpha}}.$$

Comparing (22) and (23) we see that indeed $R_{w_0} = S_{w_0}$, as desired. This completes the proof of Theorem 4.

6. WHITTAKER FUNCTIONS

We conclude this paper by showing how to compute Whittaker functions on certain metaplectic groups using the Demazure and Demazure–Lusztig operators. This follows from results in [McN, Section 15], which relate Whittaker functions to the local factors of Weyl group multiple Dirichlet series constructed in [CG10]. Since these factors can be constructed using Theorems 3 or 4, we obtain an alternative description of the Whittaker functions in the spirit of Demazure’s character formula.

Before we can define the Whittaker function of interest, we must introduce notation and quickly recall the construction of unramified principal series representations on metaplectic groups. Our presentation is taken from [McN11, McN], and the reader should look there for more details.

Let F be a local field containing the n -th roots of unity, μ_n . We choose once and for all an identification of μ_n with the complex n -th roots of unity. Let \mathcal{O} denote the ring of integers and \mathfrak{p} the maximal ideal of \mathcal{O} with uniformizer ϖ . Let q denote the order of the residue field \mathcal{O}/\mathfrak{p} . We assume that $q \equiv 1 \pmod{2n}$, so that in fact F contains the $2n$ -th roots of unity.

In order to define *Gauss sums*, we introduce an additive character ψ_F on F with conductor \mathcal{O} . Further let $(\cdot, \cdot) = (\cdot, \cdot)_{F,n}: F^\times \times F^\times \rightarrow \mu_n(F)$ be the n -th order Hilbert symbol. It is a bilinear form on F^\times that defines a nondegenerate bilinear form on $F^\times/F^{\times n}$ and satisfies

$$(x, -x) = (x, y)(y, x) = 1, \quad x, y \in F^\times.$$

Our assumption that -1 is an n -th root of unity further implies that $(\varpi, -1) = 1$. Then we define

$$(24) \quad g_i = \sum_{u \in \mathcal{O}^\times/(1+\mathfrak{p})} (u, \varpi^i) \psi_F(-\varpi^{-1}u).$$

In particular g_i depends only on the residue class of $i \pmod{n}$, $g_0 = -1$ and $g_i g_{n-i} = q$.

Now let G be a connected reductive group over F . We assume that G is unramified and split and hence arises as the base change of a Chevalley group scheme \mathbf{G} defined over \mathbb{Z} . Let $K = \mathbf{G}(\mathcal{O})$ be a maximal compact subgroup. Let T be a maximal split torus and let Λ be its group of cocharacters. Let B be a Borel subgroup containing T , let U be the unipotent radical of B , and let U^- be the opposite subgroup to U . Let Φ be the roots of T in G , and let $\Delta \subset \Phi$ be the simple roots. The Weyl group W of Φ acts on Λ , and as in Section 2 we

fix a W -invariant integer-valued quadratic form on Λ , and use it to define the sublattice $\Lambda_0 \subset \Lambda$ as in (3).

Let \tilde{G} be an n -fold metaplectic cover of G , as defined in [McN, Section 2]. Thus we have an exact sequence

$$(25) \quad 1 \rightarrow \mu_n \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

where μ_n is the group of n -th roots of unity. We choose an identification of μ_n with the complex n -th roots of unity. Denote the inverse image of any subgroup $J \subset G$ with a tilde: \tilde{J} . It is known that (25) splits canonically over U and U^- . In general (25) does not split over K , but our assumption on q implies that it does. We therefore fix a splitting $\tilde{K} \simeq \mu_n \times K$ and identify K with its image in \tilde{G} . Let H be the centralizer in \tilde{T} of $T \cap K$. The lattice Λ (respectively Λ_0) can be identified with $\tilde{T}/(\mu_n \times (T \cap K))$ (resp., $H/(\mu_n \times (H \cap K))$). Our assumptions on G imply that H is abelian, and in fact $H/(T \cap K) \simeq \mu_n \times \Lambda_0$ (although not canonically). Moreover, we may choose a lift of Λ into \tilde{G} ; we denote this lift by $\lambda \mapsto \varpi^\lambda$.

The unramified principal series representations are parametrized by complex-valued characters χ of Λ_0 . Given such a character, we obtain a character of H using the surjection $H \rightarrow \mu_n \times \Lambda_0$, where we let the roots of unity act faithfully. We induce this character to \tilde{T} and obtain a representation $(\pi_\chi, i(\chi))$. The *unramified principal series* representation $(\pi_\chi, I(\chi))$ is formed using normalized induction of this representation to \tilde{G} . More precisely, we have

$$I(\chi) = \{f: \tilde{G} \rightarrow i(\chi) : f(bg) = \delta^{1/2}(b)\pi_\chi(b)f(g), b \in \tilde{B}, g \in \tilde{G}, f \text{ locally constant}\},$$

where δ is the modular quasicharacter of \tilde{B} , and where \tilde{G} acts on $I(\chi)$ by right translation. One proves that $I(\chi)^K$ is one-dimensional; a nonzero element ϕ_K in this space of invariants is called a *spherical vector*.

Let $\psi: U^- \rightarrow \mathbb{C}$ be an unramified character. By definition this means that the restriction of ψ to each of the root subgroups $U_{-\alpha}$, $\alpha \in \Delta$ is a character of $U_{-\alpha} \simeq F$ with conductor \mathcal{O} . Then the function $\tilde{G} \rightarrow i(\chi)$ defined by

$$(26) \quad g \mapsto \int_{U^-} \phi_K(ug)\psi(u)du$$

is the *$i(\chi)$ -valued Whittaker function with character ψ* . We will obtain a complex-valued Whittaker function by applying a linear functional $\xi \in i(\chi)^*$ to the right of (26). We now explain how to construct certain functionals so that we can arrive at a very explicit formula. To do this we must be very careful about normalizations.

Recall that $\phi_K \in I(\chi)^K$ is our spherical vector. It turns out that we have an isomorphism $I(\chi)^K \simeq i(\chi)^{\tilde{T} \cap K}$ given by $f \mapsto f(1)$. Let $v_0 = \phi_K(1)$. Let A be a set of coset representatives for \tilde{T}/H ; our assumptions imply that we can assume they each have the form ϖ^λ for some $\lambda \in \Lambda$. The vectors $\{\pi_\chi(a)v_0 : a \in A\}$ give a basis of $i(\chi)$.

Now let $\tilde{\chi}: \tilde{T} \rightarrow \mathbb{C}$ be an extension of χ to \tilde{T} satisfying $\tilde{\chi}(th) = \tilde{\chi}(t)\chi(h)$ for all $t \in \tilde{T}, h \in H$. Such an extension determines a functional $\xi_{\tilde{\chi}} \in i(\chi)^*$ by

$$\xi_{\tilde{\chi}}(\pi_\chi(a)v_0) = \tilde{\chi}(a).$$

Since each a has the form ϖ^λ for $\lambda \in \Lambda$, we may write instead $\tilde{\chi}(\lambda)$ for $\tilde{\chi}(a)$. Then the complex-valued Whittaker function we want to compute is

$$(27) \quad \mathcal{W} = \mathcal{W}_{\tilde{\chi}} : g \mapsto \xi_{\tilde{\chi}} \left(\int_{U^-} \phi_K(ug)\psi(u)du \right).$$

The fact that \mathcal{W} satisfies

$$\mathcal{W}(\zeta ugk) = \zeta \psi(u) \mathcal{W}(g), \quad \zeta \in \mu_n, u \in U, g \in \tilde{G}, k \in K$$

together with the Iwasawa decomposition $G = UTK$ implies that it suffices to compute \mathcal{W} on \tilde{T} .

We are almost ready to evaluate \mathcal{W} on \tilde{T} in terms of our Demazure operators. Set $v = q^{-1}$ in the group action (7) and interpret the Weyl group action on Λ as acting on $\tilde{\chi}$ via the identification $\tilde{\chi}(\varpi^\lambda) = x^\lambda$. Further define

$$(28) \quad c_{w_0}(x) = \frac{\prod_{\alpha \in \Phi^+} (1 - q^{-1} x^{m(\alpha)\alpha})}{\prod_{\alpha \in \Phi^+} (1 - x^{m(\alpha)\alpha})}$$

Then we have the following formula of McNamara:

Theorem 15. [McN, Theorem 15.2] *Let λ be a dominant coweight. Then*

$$(\delta^{-1/2} \mathcal{W}_{\tilde{\chi}})(\varpi^\lambda) = c_{w_0}(x) \sum_{w \in W} \operatorname{sgn}(w) \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha} w(x^{w_0\lambda}),$$

where w acts on x^λ as in (7).

Actually [McN, Theorem 15.2] is written in terms of a slightly different group action introduced in [CG10], but relating the two actions leads to the statement above. Combining the previous result with Theorems 3 and 4 we arrive at our objective of expressing the Whittaker function in terms of the Demazure and Demazure–Lusztig operators.

Theorem 16. *For λ a dominant coweight,*

$$\begin{aligned} (\delta^{-1/2} \mathcal{W}_{\tilde{\chi}})(\varpi^\lambda) &= \prod_{\alpha \in \Phi^+} (1 - q^{-1} x^{m(\alpha)\alpha}) \mathcal{D}_{w_0}(x^{w_0\lambda}) \\ &= \sum_{w \in W} \mathcal{T}_w(x^{w_0\lambda}). \end{aligned}$$

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