

# LITTELmann PATTERNS AND WEYL GROUP MULTIPLE DIRICHLET SERIES OF TYPE $D$

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**ABSTRACT.** We formulate a conjecture for the local parts of Weyl group multiple Dirichlet series attached to root systems of type  $D$ . Our conjecture is analogous to the description of the local parts of type  $A$  series given by Brubaker, Bump, Friedberg, and Hoffstein [3] in terms of Gelfand–Tsetlin patterns. Our conjecture is given in terms of patterns for irreducible representations of even orthogonal Lie algebras developed by Littelmann [13].

## 1. INTRODUCTION

We begin with some notation. Let  $\Phi$  be a reduced root system of rank  $r$  and  $n$  a positive integer. Let  $F$  be a number field containing the  $2n$ -th roots of unity. Let  $S$  be a set of places of  $F$  containing the archimedean places and those that ramify over  $\mathbb{Q}$ , as well as sufficiently many more places to ensure that the ring of  $S$ -integers  $\mathcal{O}_S$  has class number 1. Let  $\mathbf{m} = (m_1, \dots, m_r)$  be a fixed nonzero tuple of elements of  $\mathcal{O}_S$ . Let  $\mathbf{s} = (s_1, \dots, s_r)$  be an  $r$ -tuple of complex variables.

Given the data above, one can form a *Weyl group multiple Dirichlet series*. This is a Dirichlet series in the  $r$  variables  $s_i$  with a group of functional equations isomorphic to the Weyl group  $W$  of  $\Phi$ . More precisely, one can define a set of functions of the form

$$Z(\mathbf{s}; \mathbf{m}, \Psi) = Z_{\Phi}^n(\mathbf{s}; \mathbf{m}, \Psi) = \sum_{\mathbf{c}} \frac{H(\mathbf{c}; \mathbf{m}) \Psi(\mathbf{c})}{\prod |c_i|^{s_i}},$$

where each  $c_i$  ranges over nonzero elements of  $\mathcal{O}_S$  modulo units,  $\Psi$  is taken from a certain finite-dimensional complex vector space  $\Omega$  of functions on  $(F_S^\times)^r$ , and  $H$  is an important function we shall say more about shortly. Then the collection of all such  $Z$  as  $\Psi$  ranges over a basis of  $\Omega$  satisfies a group of functional equations isomorphic to  $W$  with an appropriate scattering matrix. For more about why Weyl group multiple Dirichlet series are interesting objects, as well as a discussion about the basic framework for their construction, we refer to [6].

The heart of the construction of  $Z$  is the function  $H$ . This function must be carefully defined to ensure that  $Z$  satisfies the correct group of

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functional equations. The heuristic of [6] dictates how to define  $H$  on the *powerfree* tuples  $\mathbf{c}, \mathbf{m}$  (those tuples such that the product  $c_1 \cdots c_r m_1 \cdots m_r$  is squarefree). Moreover, it is further specified in [6] how the values of  $H$  on the *prime power* tuples  $\mathbf{c} = (\varpi^{k_1}, \dots, \varpi^{k_r})$ ,  $\mathbf{m} = (\varpi^{l_1}, \dots, \varpi^{l_r})$ , where  $\varpi \in \mathcal{O}_S$  is a prime, determine  $H$  on all tuples.

Thus, writing  $\ell$  for a tuple of nonnegative integers  $(l_1, \dots, l_r)$  and letting  $\varpi^\ell$  denote the tuple  $(\varpi^{l_1}, \dots, \varpi^{l_r})$ , the construction of  $Z$  reduces to understanding the multivariate generating function

$$(1) \quad N(x_1, \dots, x_r; \ell) := \sum_{k_i \geq 0} H(\varpi^{k_1}, \dots, \varpi^{k_r}; \varpi^\ell) x_1^{k_1} \cdots x_r^{k_r}.$$

At present there are two different approaches to understanding the generating function (1), and thus to constructing Weyl group multiple Dirichlet series. Both are related to characters of representations of the semisimple complex Lie algebra attached to  $\Phi$ . Let  $\omega_i, i = 1, \dots, r$  be the fundamental weights of  $\Phi$  and let  $\theta$  be the strictly dominant weight  $\sum (l_i + 1)\omega_i$ .

- The *Gelfand–Tsetlin* approach [2,3,5], which works for  $\Phi = A_r$ , gives formulas for the coefficients  $H(\varpi^{k_1}, \dots, \varpi^{k_r}; \varpi^\ell)$ . These formulas are written in terms of Gauss sums and statistics extracted from Gelfand–Tsetlin patterns for the representation of  $\mathfrak{sl}_{r+1}(\mathbb{C})$  of lowest weight  $-\theta$ .
- The *averaging* approach [8–10, 12], which works for all  $\Phi$ , uses a “metaplectic” deformation of the Weyl character formula to construct a rational function with known denominator, whose numerator is then taken to define  $N$ .

Both approaches have their advantages and limitations. The Gelfand–Tsetlin construction gives very explicit formulas for  $H$ , formulas that (remarkably) are uniform in  $n$  and that lead to a direct connection with the global Fourier coefficients of Borel Eisenstein series on the  $n$ -fold cover of  $\mathrm{SL}_{r+1}$  [1], but suffers from the obvious disadvantage that it only works for type  $A$ . The averaging approach, on the other hand, works for all  $\Phi$ , quickly leads to the definition of  $Z$ , yet has the drawback that it seems difficult to get similarly explicit formulas for the coefficients of  $N$ . By combining recent work of Chinta–Offen [11] and McNamara [14], we know that in type  $A$  the two definitions of  $N$  coincide, although it seems difficult to give a direct combinatorial proof.

This note arose from our attempts to understand the Gelfand–Tsetlin approach to (1). In the course of studying [3], it became plain to us that the most suitable language to understand the constructions in [3] is that of Kashiwara’s *crystal graphs*, as encoded in the generalization of the Gelfand–Tsetlin basis due to Littelmann [13], which we call *Littelmann patterns*. Indeed, the definitions in [3] become much more transparent when phrased in terms of these patterns.

To test the relevance of this observation, we decided to try to formulate a Littelmann analogue of the Gelfand–Tsetlin construction when  $\Phi$  is a root

system of type  $D$ . The main result of this note is thus Conjecture 1, which explicitly describes the generating function  $N(x_1, \dots, x_r; \ell)$  for the  $\varpi$ -part of the type  $D$  Weyl group multiple Dirichlet series constructed using the averaging method. We remark that for  $n = 1$ , Conjecture 1 gives a type  $D$  analogue of a theorem of Tokuyama [15].

We have some limited evidence for the truth of Conjecture 1.

First, for  $D_2 \simeq A_1 \times A_1$ , the conjecture is easily seen to be true.

Next, we have tested the conjecture for  $D_3$  when  $n \leq 4$  and for  $D_4$  when  $n \leq 2$ , by computing the  $\varpi$ -parts by averaging for many tuples  $\ell$  and comparing with the predictions of Conjecture 1. In all cases there was complete agreement. Note that  $D_3 \simeq A_3$ , so the  $\varpi$ -part of the  $D_3$ -series has already been described explicitly using the results of [3], and in this guise has already been compared extensively with  $\varpi$ -parts constructed by averaging. Nevertheless, agreement in rank 3 between  $\varpi$ -parts constructed using Conjecture 1 and using averaging is a nontrivial check, since  $D_3$  Littelmann patterns are quite different from  $A_3$  Gelfand–Tsetlin patterns.

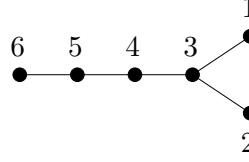
Finally, recently Brubaker and Friedberg have computed the global Whittaker coefficients of Eisenstein series on covers of  $\mathrm{GL}_4$  by inducing from the parabolic subgroup of type  $\mathrm{GL}_2 \times \mathrm{GL}_2$  [4]. Their computations—which build on earlier work of Bump–Hoffstein [7] and are the first attempts to extend the results of [1] beyond type  $A$  and to work with other parabolic subgroups—express the Whittaker coefficients in terms of certain exponential sums. In the course of their work Brubaker and Friedberg found that the integrals can be broken up in accordance with the decomposition of  $H(\varpi^{k_1}, \dots, \varpi^{k_r}; \varpi^\ell)$  given by Conjecture 1, and that if one does so the contributions to the global Whittaker coefficient exactly agrees with Conjecture 1. We find this connection between Eisenstein series and  $\varpi$ -parts to be strongly convincing evidence of the correctness of Conjecture 1.

## 2. LITTELmann PATTERNS

Let  $\mathfrak{g}$  be the simple complex Lie algebra of type  $D_r$ , in other words the Lie algebra of the group  $\mathrm{SO}_{2r}(\mathbb{C})$ . Let  $\theta$  be a dominant weight of  $\mathfrak{g}$  and let  $V_\theta$  be the irreducible  $\mathfrak{g}$ -module of highest weight  $\theta$ . In [13, §7] Littelmann describes a way to index a basis of  $V_\theta$  using patterns that are analogous to the classical Gelfand–Tsetlin patterns for the Lie algebra of  $\mathrm{SL}_r(\mathbb{C})$ . In this section we recall his construction.

First we label vertices of the Dynkin diagram of  $\mathfrak{g}$  with the integers  $1, \dots, r$ . We label the upper node of the right prong 1, the lower node of the prong 2, the node at the elbow of the prong 3, and then the remaining nodes increase from 4 to  $r$ , reading right to left (Figure 1). We remark that this is not the standard labelling by Bourbaki, which begins with 1 at the left of the diagram.

A pattern  $T$  for  $D_r$  consists of a collection of integers  $a_{i,j}$ , where  $1 \leq i \leq r-1$  and  $i \leq j \leq 2r-2$ . We picture  $T$  by drawing the integers placed

FIGURE 1. The diagram for  $D_6$ 

in  $r - 1$  rows of centered boxes. The first row contains  $2r - 2$  boxes, the second  $2r - 4$  boxes, and so on down to the  $(r - 1)$ -st row, which contains 2 boxes. The integers are placed in the boxes so that  $a_{i,i}$  is placed in the leftmost box of the  $i$ th row, and then the remaining integers  $a_{i,j}$  are put in the boxes in order as  $j$  increases. We define an involution on each row by  $\bar{a}_{i,j} = a_{i,2r-1-j}$ .

To index a weight vector in  $V_\theta$ , there are two sets of inequalities the  $a_{i,j}$  must satisfy. The first is independent of  $\theta$ : in each row we must have

(2)  $a_{i,i} \geq a_{i,i+1} \geq \cdots \geq a_{i,r-2} \geq a_{i,r-1}, a_{i,r} \geq a_{i,r+1} \geq \cdots \geq a_{i,2r-1-i} \geq 0$ ,  
or, using the bar notation,

$$a_{i,i} \geq a_{i,i+1} \geq \cdots \geq a_{i,r-2} \geq a_{i,r-1}, \bar{a}_{i,r-1} \geq \bar{a}_{i,r-2} \geq \cdots \geq \bar{a}_{i,i} \geq 0$$

In other words, the  $a_{i,j}$  are weakly decreasing in the rows, with the exception that no comparison is made between  $a_{i,r-1}$  and  $a_{i,r}$ . Both of these entries, however, are required to be  $\leq a_{i,r-2}$  and  $\geq a_{i,r+1}$ .

**Definition 1.** A pattern  $T$  is *admissible* if  $T$  satisfies (2) for all  $i$ .

Figure 2 shows an admissible pattern for  $D_4$ .

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 6 | 6 | 5 | 4 | 4 | 2 |
| 3 | 3 | 2 | 1 |   |   |
| 1 | 3 |   |   |   |   |

FIGURE 2.

The next set of inequalities involves the highest weight  $\theta$ . Write

$$\theta = \sum m_k \omega_k,$$

where the  $\omega_k$  are the fundamental weights. Then an admissible  $T$  will correspond to a weight vector in  $V_\theta$  if  $T$  satisfies

(3)  $\bar{a}_{i,j} \leq m_{r-j+1} + s(\bar{a}_{i,j-1}) - 2s(a_{i-1,j}) + s(a_{i-1,j+1}) \quad \text{for } j \leq r-2,$   
 (4)  $a_{i,j} \leq m_{r-j+1} + s(a_{i,j+1}) - 2s(\bar{a}_{i,j}) + s(\bar{a}_{i,j-1}) \quad \text{for } j \leq r-2,$   
 (5)  $a_{i,r-1} \leq m_2 + s(\bar{a}_{i,r-2}) - 2t(a_{i-1,r-1}), \text{ and}$   
 (6)  $a_{i,r} \leq m_1 + s(\bar{a}_{i,r-2}) - 2t(a_{i-1,r}),$

where we write for  $j < r - 1$

$$\begin{aligned} s(\bar{a}_{i,j}) &= \bar{a}_{i,j} + \sum_{k=1}^{i-1} (a_{k,j} + \bar{a}_{k,j}), \\ s(a_{i,j}) &= \sum_{k=1}^i (a_{k,j} + \bar{a}_{k,j}), \\ s(a_{i,r-1}) &= s(\bar{a}_{i,r-1}) = \sum_{k=1}^i a_{k,r-1} + a_{k,r}, \\ t(a_{i,r-1}) &= \sum_{k=1}^i a_{k,r-1}, \quad t(a_{i,r}) = \sum_{k=1}^i a_{k,r}. \end{aligned}$$

**Definition 2.** A pattern  $T$  is  $\theta$ -admissible if  $T$  is admissible and its entries satisfy (3)–(6).

Note that the inequalities for the  $i$ th row only involve the entries of  $T$  on the  $i$ th and  $(i-1)$ st rows. Moreover when ordered in terms of increasing  $i$ , there is a unique inequality in which a given entry  $a_{i,j}$  appears on the left.

**Definition 3.** We say that an entry in a  $\theta$ -admissible pattern is *critical* if this first inequality is actually an equality.

To complete our discussion of Littelmann patterns, we must assign a weight  $\lambda(T)$  to each pattern  $T$ . This is a vector  $\lambda(T) = (\lambda_1, \dots, \lambda_r)$  of nonnegative integers, where

$$\lambda_k = \begin{cases} \sum_{i=1}^{r-1} (a_{i,r+1-k} + \bar{a}_{i,r+1-k}) & k = 3, 4, \dots, r \\ \sum_{i=1}^{r-1} a_{i,r-2+k} & k = 1, 2. \end{cases}$$

We write  $|\lambda| = \lambda_1 + \dots + \lambda_r$ . In our conjecture, if a pattern  $T$  occurs for the twist  $\theta = \sum m_i \omega_i$ , it will contribute to the coefficient of  $\mathbf{x}^{\lambda(T)} := x_1^{\lambda_1} \cdots x_r^{\lambda_r}$  in the numerator  $N(\mathbf{x}, \ell)$ , where  $\ell = (l_1, \dots, l_r)$  and  $l_i = m_i - 1$ . For instance, the pattern in Figure 2 contributes to the coefficient of  $x_1^9 x_2^9 x_3^{14} x_4^8$ , with  $x_1$  corresponding to left middle column of three entries and  $x_2$  to the right middle column of three entries.

### 3. THE DECORATED GRAPH OF A PATTERN

Let  $T$  be a  $\theta$ -admissible pattern. We want to associate to  $T$  a graph  $\Gamma(T)$ . The graph  $\Gamma(T)$  will also potentially be endowed with *decorations*, which will be circled vertices. The vertices of  $\Gamma(T)$  correspond to the entries of  $T$ ; the graph will have at least one connected component for each row of  $T$ .

We begin by describing how each row determines a subgraph. Consider the  $i$ th row of  $T$ . Each entry  $a_{i,j}$  in this row gives a vertex. We draw the corresponding vertices in a row, with the two vertices in the middle corresponding to the incomparable entries  $a_{i,r-1}, a_{i,r}$  entries arranged vertically.

For definiteness we assign  $a_{i,r-1}$  to the top vertex and  $a_{i,r}$  to the bottom vertex. See Figure 3 for the arrangement for the top row of a pattern for  $D_6$ .

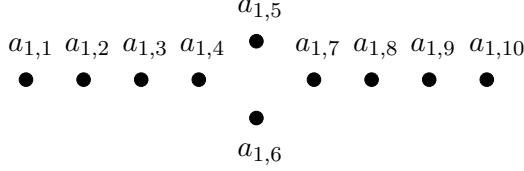


FIGURE 3. The vertices for the top row of  $D_6$

Now join two vertices by an edge if they appear consecutively in the inequalities (2), are equal, and are comparable in (2). Note that we do not join the vertices corresponding to  $a_{i,r-1}, a_{i,r}$  by an edge if they happen to be equal, since they are not comparable in (2). This gives a graph for this row. We then do the same for each row of  $T$ . The result is  $\Gamma(T)$  without decorations.

Certain symmetric connected components that arise in the construction of  $\Gamma(T)$  will play a special role in our conjecture:

**Definition 4.** Let  $T$  be an admissible pattern and suppose  $a_{i,j} = \bar{a}_{i,j}$  for some  $i, j$  with  $j \neq r-1, r$ . Then the component of  $\Gamma(T)$  containing  $a_{i,j}, \bar{a}_{i,j}$  is called a *multiple leaner*. If in addition  $a_{i,j-1} \neq a_{i,j}$  and  $\bar{a}_{i,j-1} \neq \bar{a}_{i,j}$  then we say the multiple leaner is *symmetric*. We define the *length*  $l(C)$  of a symmetric multiple leaner to be half the number of its vertices.

The term *leaning* is inspired by [3]; see also §5. Figure 4 shows an example of a symmetric multiple leaner of length 5, when all the entries in the top row of a pattern for  $D_6$  are equal. Note that the minimal length of a symmetric multiple leaner is 2, and that multiple leaners can appear in patterns for  $D_3$ , but not for  $D_2$ .

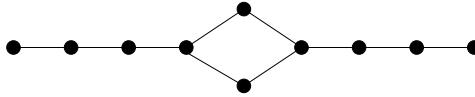


FIGURE 4.

To complete the construction of  $\Gamma(T)$  we must describe how to add the decorations. This is very simple: we circle each vertex whose corresponding entry is critical in the sense of Definition 3.

Figure 5 shows an example of building the decorated graph of the Littelmann pattern in Figure 2. We assume that a highest weight  $\theta$  has been specified so that the circled vertices in the graph correspond to critical entries.

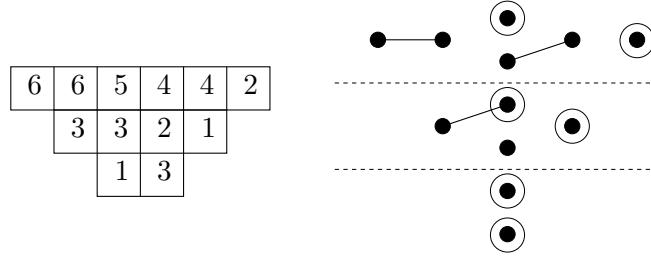


FIGURE 5.

## 4. STRICTNESS

In the following for  $k = 1, \dots, n-1$  we write  $g_k$  for the Gauss sum  $g(\varpi^{k-1}, \varpi^k)$  (see for example [8] for the definition of the Gauss sums). For convenience we extend the notation and define  $g_0$  to be  $-1$ . It is also convenient to define  $g_m$  for  $m \geq n$  by  $g_m = g_k$ , where  $m = k \bmod n$  and  $k = 0, \dots, n-1$ . We let  $p$  be the norm of  $\varpi$ .

In [3] certain patterns for a given weight are discarded and do not contribute to the relevant coefficient of  $N$ ; such patterns are called *strict* in [3]. In type  $A$  strictness corresponds to an easily stated property for Gelfand–Tsetlin patterns. If one interprets the definition of strictness in [3] in terms of type  $A$  Littelmann patterns, one sees that a type  $A$  pattern is nonstrict exactly when

- an entry is simultaneously 0 and critical, or
- there are two adjacent entries that are equal, with the left entry critical.

We take these to be our definition for type  $D$  patterns as well:

**Definition 5.** A type  $D$  Littelmann pattern  $T$  is called *strict* if the following conditions hold:

- No component of  $\Gamma(T)$  contains a vertex with a circled 0.
- No component of  $\Gamma(T)$  that is not a multiple-leaner contains a subgraph of the form shown in Figure 6 (in this figure, the rightmost vertex is less than the left vertex in the partial order from (2)).

Note that the subgraph from Figure 6 is allowed to appear in multiple-leaners.



FIGURE 6.

## 5. LEANING AND STANDARD CONTRIBUTIONS

Let  $T$  be a strict pattern, and let  $\Gamma(T)$  be the associated decorated graph. For any connected component  $C$  of  $\Gamma(T)$ , let  $y_C$  be the *rightmost* vertex, in the sense of the order induced by the inequalities (2). If  $C$  has two rightmost vertices, meaning that it is in the  $i$ -th row and contains entries  $a_{i,r-2} = a_{i,r-1} = a_{i,r} \neq a_{i,r+1}$ , then we define the rightmost vertex to be the vertex corresponding to  $a_{i,r-1}$ , that is, the upper vertex in Figure 3.

**Definition 6.** Let  $T$  be a pattern and  $\Gamma = \Gamma(T)$  the associated decorated graph. Fix  $n$  and let  $y$  be an entry of  $T$ . We define the *standard contribution*  $\sigma(y)$  by the following rule:

- If the vertex corresponding to  $y \neq 0$  is uncircled, then we put  $\sigma(y) = 1 - 1/p$  if  $n$  divides  $y$  and  $\sigma(y) = 0$  otherwise.
- If the vertex corresponding to  $y \neq 0$  is circled, then we put  $\sigma(y) = g_k/p$ , where  $y = k \bmod n$  and  $k = 0, \dots, n-1$ .

Note that  $\sigma(y)$  depends on  $n$  and  $\theta$ , even though we omit them from the notation.

We are almost ready to state our conjecture. There is one more phenomenon that plays a role, namely *leaning*. Essentially, leaning means that if entries are consecutive and equal in a Littelmann pattern  $T$ , where consecutive means adjacent in (2), then only one should contribute to the corresponding coefficient of  $N(\mathbf{x}; \ell)$ . This is why we introduce the graph  $\Gamma(T)$ . Its connected components keep track of these equalities among entries.

Thus we are led to consider contributions of the connected components of  $\Gamma(T)$ , not just the entries. There is further slight twist that the contribution of a multiple leaning component is different from that of all other components:

**Definition 7.** Let  $C$  be a connected component of  $\Gamma(T)$ . The *standard contribution*  $\sigma(C)$  of  $C$  is defined as follows:

- If  $C$  is not a multiple leaner, then we put  $\sigma(C) = \sigma(y_C)$ , where  $y_C$  is the rightmost entry of  $C$ .
- If  $C$  is a multiple leaner that is not symmetric, let  $y_C$  be the entry on the endpoint of its shorter leg. Then we define  $\sigma(C) = \sigma(y_C)$ .
- If  $C$  is a symmetric multiple leaner, then let  $y_C$  be its rightmost entry  $a_{i,j}$  and  $v_C$  (upsilon = Greek  $y$ ) to be the entry  $a_{i,j-1}$ . Then we define

$$\sigma(C) = \begin{cases} \sigma(y_C)(1 - 1/p^{l(C)}) & \text{if } y_C \text{ is uncircled,} \\ \sigma(y_C)\sigma(v_C)(1/p^{l(C)-1}) & \text{if } y_C \text{ is circled,} \end{cases}$$

where  $l(C)$  is defined to be half the number of vertices of  $C$  (Definition 4).

We are now ready to state our conjecture:

**Conjecture 1.** Let  $N(\mathbf{x}; \ell) = \sum_{\lambda} a_{\lambda} \mathbf{x}^{\lambda}$  be the  $\varpi$ -part constructed by averaging [8, 10] for the Weyl group multiple Dirichlet series  $Z_{\Phi}^n(\mathbf{s}; \mathbf{m}, \Psi)$ . Then we have

$$(7) \quad a_{\lambda} = p^{|\lambda|} \sum_T \prod_{C \subset \Gamma(T)} \sigma(C),$$

where the sum is taken over all strict patterns  $T$  of weight  $\lambda$  and with highest weight  $\theta = \sum (l_i + 1)\omega_i$ , and the product is taken over the connected components of  $\Gamma(T)$ .

**Example 1.** Suppose the pattern in Figure 2 appears for a highest weight  $\theta$  such that the decorated graph appears in Figure 5. Suppose  $n = 2$ . Then the contribution of this pattern to the coefficient of  $x_1^9 x_2^9 x_3^{14} x_4^8$  will be

$$p^{40} \left(1 - \frac{1}{p}\right)^3 \left(-\frac{1}{p}\right) \left(\frac{g_1}{p}\right)^5.$$

**Example 2.** We consider another example for  $n = 2$ . Suppose the twisting parameter is  $\ell = (0, 1, 2, 0)$ , which corresponds to the highest weight  $\theta = \omega_1 + 2\omega_2 + 3\omega_3 + \omega_4$ . We will compute the coefficient  $a_{\lambda}$  of the monomial  $\mathbf{x}^{\lambda} = x_1^{10} x_2^{10} x_3^{17} x_4^{10}$ . Note that  $|\lambda| = 47$ .

There are 27 Littelmann patterns that we must consider. Six of these patterns are nonstrict, for instance the pattern shown in Figure 7. Of the remaining 21, only 2 give nonzero contributions; these patterns  $T_1, T_2$  appear in Figures 8–9. Note that Figure 9 contains a multiple leaner of length 2. All of the other 19 patterns have an odd entry that is not circled, and thus have a connected component in  $\Gamma(T)$  with standard contribution equal to zero.

Each vertex in  $\Gamma(T_1)$  is its own connected component. We see 3 uncircled even nonzero entries, 5 circled even nonzero entries, and 3 circled odd entries. Thus  $T_1$  contributes

$$p^{47} \left(1 - \frac{1}{p}\right)^3 \left(-\frac{1}{p}\right)^5 \left(\frac{g_1}{p}\right)^3$$

to  $a_{\lambda}$ .

The pattern  $T_2$  has a multiple leaner  $C$  of length  $l(C) = 2$ . Its rightmost entry  $y_C$  is circled, and the entry  $v_C$  is uncircled. We have  $\sigma(y_C) = -1/p$ ,  $\sigma(v_C) = (1 - 1/p)$ . These appear in (7) multiplied by the additional factor  $1/p$  to account for the length of  $C$ . Each of the remaining vertices is its own connected component, and we have no uncircled nonzero evens, 4 circled nonzero evens, and 3 circled odds. Thus  $T_2$  contributes

$$p^{47} \left(-\frac{1}{p}\right)^4 \left(\frac{g_1}{p}\right)^3 \left(-\frac{1}{p}\right) \left(1 - \frac{1}{p}\right) \left(\frac{1}{p}\right)$$

to  $a_{\lambda}$ . After simplifying we find

$$a_{\lambda} = -p^{36} (p^3 - 2p^2 + 2p - 1) g_1^3,$$

in agreement with the  $\varpi$ -part from [10].

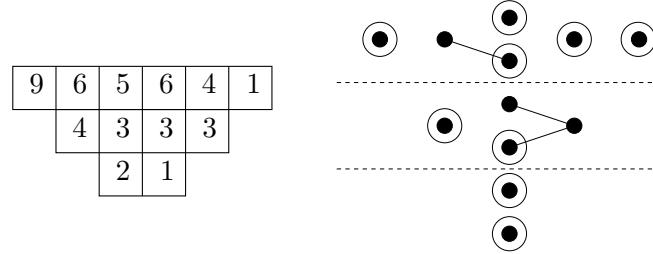
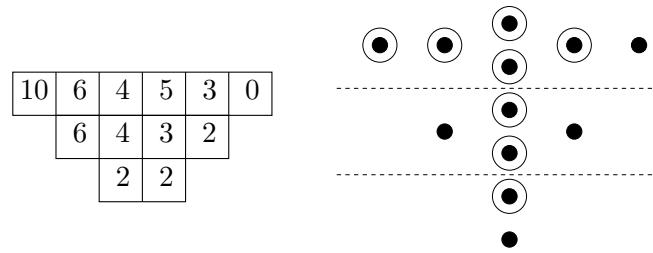
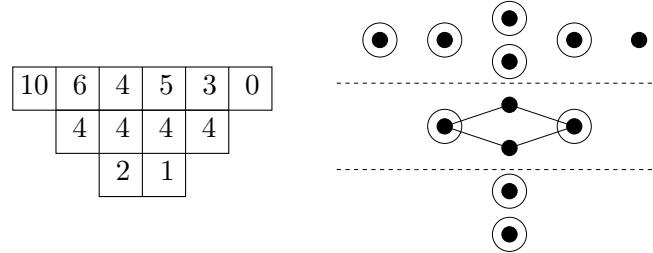


FIGURE 7. A nonstrict pattern

FIGURE 8.  $T_1$ FIGURE 9.  $T_2$ 

**Example 3.** We conclude by describing an example for  $D_4$  when  $n = 1$ . We put  $\ell = (0, 0, 0, 0)$  (the “untwisted” case), so that the highest weight is  $\omega_1 + \omega_2 + \omega_3 + \omega_4$ . The polynomial  $N(\mathbf{x}; \ell)$  is supported on 601 monomials. There are 4096 Littelmann patterns to consider, 2216 of which are nonstrict. The remaining patterns each give a nonzero contribution to  $N$ . The resulting polynomial can be written succinctly as

$$N(\mathbf{x}; \ell) = \prod_{\alpha > 0} (1 - p^{d(\alpha)-1} \mathbf{x}^\alpha),$$

where the product is taken over the positive roots. Here  $d(\alpha) = k_1 + k_2 + k_3 + k_4$  if  $\alpha$  is the linear combination of simple roots  $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4$ , and  $\mathbf{x}^\alpha$  refers to the monomial  $x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4}$ .

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