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Multiple Dirichlet Series and Automorphic Forms

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Abstract. This article gives an introduction to the multiple Dirichlet series arising from sums of twisted automorphic L-functions. We begin by explaining how such series arise from Rankin-Selberg constructions. Then more recent work, using Hartogs' continuation principle as extended by Bochner in place of such constructions, is described. Applications to the nonvanishing of L-functions and to other problems are also discussed, and a multiple Dirichlet series over a function field is computed in detail.

1. Motivation

Of the major open problems in modern mathematics, the Riemann hypothesis, which states that the nontrivial zeroes of the Riemann zeta function $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$, is one of the deepest and most profoundly important. A consequence of the Riemann Hypothesis which has far reaching applications is the Lindelöf Hypothesis. This states that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that for all t,

$$|\zeta(1/2 + it)| < C(\epsilon)|t|^{\epsilon}.$$

The Lindelöf Hypothesis remains as unreachable today as it was 100 years ago, but there has been a great deal of progress in obtaining approximations of it. These are results of the form $|\zeta(1/2 + it)| < C(\epsilon)|t|^{\kappa+\epsilon}$, where $\kappa > 0$ is some fixed real number. For example, Riemann's functional equation for the zeta function, together with Stirling's approximation for the gamma function and the Phragmen-Lindelöf principle, are sufficient to obtain what is known as *the convexity bound* for the zeta function, namely $\kappa = \frac{1}{4}$, or: $\left|\zeta\left(\frac{1}{2} + it\right)\right| < C(\epsilon)|t|^{\frac{1}{4}+\epsilon}$.

Any improvement over $\frac{1}{4}$ in this upper bound is known as "breaking convexity." There are also many known generalizations of $\zeta(s)$ and analogous definitions of convexity breaking that are viewed with great interest. This is, first, because of the

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connection with the Lindelöf Hypothesis, and second, because any improvement on the convexity bound or the current best value of κ tends to have dramatic consequences.

Dirichlet generalized the zeta function and introduced L-series. A well-known example is

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s},$$

where χ_d is a character of $(\mathbb{Z}/d\mathbb{Z})^\times$. These and other L-series mirror the Riemann zeta function in that they have an analytic continuation and a functional equation. They also are conjectured to satisfy a corresponding generalized Riemann Hypothesis. The presence of the extra parameter d leads naturally to the investigation of the behavior of $L(1/2 + it, \chi_d)$ for varying d, t, From this perspective, one can formulate the Lindelöf Hypothesis "in the d aspect", which states that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that for all d, $|L(1/2, \chi_d)| < C(\epsilon)|d|^\epsilon$. In a manner completely analogous to $\zeta(s)$ the functional equation for $L(s, \chi_d)$ can be used to obtain a basic convexity result: $|L(1/2, \chi_d)| < C(\epsilon)|d|^{\frac{1}{4}+\epsilon}$. The first breaking of convexity for $L(1/2, \chi_d)$ was accomplished by Burgess [**17**], with $\kappa = 3/16$, and recently there has been the result of Conrey and Iwaniec [**22**], with $\kappa = 1/6$. Such approximations to the Lindelöf Hypothesis in the d aspect have important applications to such diverse fields as mathematical physics, computational complexity, and cryptography.

The generalizations continue. One can consider, in place of $\zeta(s)$ or $L(s, \chi_d)$, the L-functions associated to automorphic forms on $GL(r)$, with extra parameters corresponding to various generalizations of χ_d. In most of these instances one expects generalizations of the Riemann and Lindelöf Hypotheses to be true and the consequences would again be remarkable.

Fortunately, if a result is elusive for a single object it is often more within reach when the same question is asked about an average over a family of similar objects. For example, consider the family of Dirichlet L-series $L(s, \chi_d)$ with χ_d quadratic (i.e. $\chi_d^2 = 1$). This family can be collected together in the *multiple Dirichlet series*

$$Z(s, w) = \sum_d \frac{L(s, \chi_d)}{d^w}.$$

where the sum ranges over, for example, discriminants of real quadratic fields. This is a very basic instance of the multiple Dirichlet series discussed in this article. It is shown in [**34**] that $Z(1/2, w)$ is absolutely convergent for $\Re w > 1$ and has an analytic continuation past $\Re w = 1$ with a pole of order 2 at the point $w = 1$. By combining this result with basic Tauberian techniques one may show that there exists a non-zero constant c such that for large X

$$\sum_{0<d<X} L(1/2, \chi_d) \sim cX \log X,$$

the sum again going over discriminants of real quadratic fields. It follows that the average value of $L(1/2, \chi_d)$ for $d < X$ takes the form of a constant times $\log X$ for d in this range and, thus, the Lindelöf Hypothesis in the d aspect is true "on average." Results of this type are significant in their own right and can also have important applications.

One of the major breakthroughs in analytic number theory in the last 5 years has been the following discovery: The assumption that the zeros of L-functions are distributed in the same way as the eigenvalues of random hermitian matrices allows one to obtain precise conjectures on the statistical distribution of values of L-functions. For example, the conjectured moments of the Riemann zeta function, by Keating and Snaith [**41**], were unattainable until the incorporation of random matrix models into the theory. A major connection between this work and multiple Dirichlet series was observed in [**25**] where it was shown that the conjectures obtained by random matrix theory could also be read off from the polar divisors of certain multiple Dirichlet series. It seems likely that multiple Dirichlet series will play a key role in the future study of the statistical distribution of L-values.

In this article we discuss generalizations of the function $Z(s, w)$ introduced above, generalizations that capture the behavior of a family of twists of an automorphic L-function. We describe different methods for obtaining the meromorphic continuations of such objects, and consequences that can be drawn from the continuations. Section 2 introduces the families of twisted L-functions of concern. It also describes a number of Rankin-Selberg constructions that give rise to double Dirichlet series. Section 3 concerns quadratic twists. We begin with a heuristic that explains why these sums of twisted L-functions should have continuation in w beyond the region of absolute convergence. We next describe the several-complex-variable methods that seem most effective in terms of continuation of the multiple Dirichlet series. We conclude with various applications, of interest both in their own right and also as illustrations of the kinds of theorems that can be established by these methods. Section 4 concerns higher order twists. The situation concerning sums of higher twists is more complicated, with Gauss sums playing a key role, and in the few known examples one is led to continue several different families of weighted series simultaneously. Once again, various applications are presented. Section 5 gives an explicit example in the function field setting, where many multiple Dirichlet series can be shown to be rational functions in several complex variables. The final section gives some additional examples and concluding remarks.

2. The Family of Twists of a Given L-Function

2.1. The basic questions. Fix an integer $n \geq 2$ and let F be a global field containing all n-th roots of unity. (The reader may choose to focus on number fields now, but in Section 5 we will give a concrete example in the function field case.) Let π be a *fixed* automorphic representation of $\mathrm{GL}(r)$ over the field F, with standard L-function

$$L(s, \pi) = \sum c(m)|m|^{-s}$$

for $\Re(s)$ sufficiently large. (In this article $L(s, \pi)$ refers to the finite part of the L-function.) Here $|m|$ denotes (an abuse of notation) an absolute norm. Throughout the paper we normalize all L-functions to have functional equation under $s \mapsto 1 - s$. Then our basic problem is to study the family of twisted L-functions

$$L(s, \pi \times \chi) = \sum c(m)\chi(m)|m|^{-s}$$

where we fix π and vary the twist by a character χ; χ will range over the idèle class characters of order *exactly* n. We may also wish to modify the problem, and suppose instead that χ ranges over the subset of idèle class characters of order exactly n with local factors χ_v specified at a finite number of places.

There are several natural questions to ask about this set of L-functions. The first is *nonvanishing*:

(1) Given a point in the critical strip s_0 (with $0 < \Re(s_0) < 1$), can one show there exist infinitely many χ as above with $L(s_0, \pi \times \chi) \neq 0$? This question goes back to Shimura [**51**], Rohrlich [**49**], and Waldspurger [**55**]. A particularly interesting choice is $s_0 = \frac{1}{2}$. For example, L-series associated to elliptic curves of positive rank will conjecturally vanish at $s_0 = \frac{1}{2}$ but twists may not.
(2) If $n = 2$ (the case of quadratic twists) and π is self-contragredient, and if $\epsilon(\frac{1}{2}, \pi \times \chi) = -1$ for all twists χ under consideration, can one show there exist infinitely many χ such that $L'(\frac{1}{2}, \pi \times \chi) \neq 0$? Note that under these hypotheses, the functional equation guarantees a zero of odd order for each twisted L-function at the center of the critical strip.

In these questions, we need not assume that π is cuspidal – indeed, $L(s, \pi)$ could be a product of L-series for lower-rank groups. Then the first question becomes that of establishing a simultaneous non-vanishing theorem. Even in the case of two independent GL(2) holomorphic modular forms, it is not known that there exists a single quadratic twist such that both twisted L-functions are nonzero at the center of the critical strip. In the case of two modular forms of weight 2, such a statement would imply that given two elliptic curves E_1, E_2 over $\mathbb{Q}$ there exists a fundamental discriminant D such that both twists E_1^D and E_2^D have finite Modell-Weil groups; this is not presently known. Using multiple Dirichlet series, in fact one can establish simultaneous non-vanishing for points s_0 in the critical strip but sufficiently far from the center of the strip [**20**] (see Theorem 6.1 in Section 6.2 below). Such results can also be proved by the large sieve inequality, but the advantage of the multiple Dirichlet series method is that the interval of nonvanishing obtained is independent of the base field.

A related question, in some sense sharper, is to ask about the *distribution* of twisted L-values. That is, one can seek to study the distribution of $L(s, \pi \times \chi)$ as we vary χ as above. For example, for positive integers k and weighting factors $a(s, \pi, d)$ we can study the asymptotics of the moments

$$\sum_{\mathrm{cond}(\chi) < \mathrm{X}} L(s, \pi \times \chi)^k a(s, \pi, d).$$

Given π and k, Langlands' theory of Eisenstein series implies that there is an isobaric automorphic representation Π_k such that $L(s, \Pi_k \times \chi) = L(s, \pi \times \chi)^k$. So it is natural to focus on the first moment, but to take π to be general. Establishing a suitable mean-value theorem for such moments would imply the Lindelöf hypothesis in the d-aspect.

Given a collection of interesting numbers $a(d)$, the idea of studying their associated Dirichlet series $\sum a(d)d^{-s}$ is well-known. In the questions above, the interesting numbers are themselves Dirichlet series: $a(d) = L(s, \pi \times \chi_d)$. Here χ_d (or $\chi_d^{(n)}$ when we need to indicate the cover) is the character given by the n-th power residue symbol $\chi_d(a) = \left(\frac{a}{d}\right)_n$, and is attached by class field theory to the extension $F(\sqrt[n]{d})/F$. Thus the sum of the numbers $L(s, \pi \times \chi_d)$ is an infinite sum of one-variable Dirichlet series—a multiple Dirichlet series. More generally, one may introduce a weighting factor $a(s,\pi,d)$ and construct

$$Z(s,w) \;=\; \sum_{d} \frac{L(s,\pi\times\chi_d)a(s,\pi,d)}{|d|^w}. \tag{2.1}$$

Such a series will converge for $\Re(s),\Re(w)$ sufficiently large. The numerators are Langlands L-functions on $GL(r)$ and so each continues individually to all complex s. Our goal is to find appropriate weighting factors $a(s,\pi,d)$ so that this series is well-behaved *in* w. Indeed, as we shall explain, in some cases weight factors exist such that the double Dirichlet series (2.1) possesses meromorphic continuation to all $(s,w)\in\mathbb{C}^2$ and moreover satisfies a finite group (typically non-abelian) of functional equations in (s,w).

In the case that $L(s,\pi)$ is a product of lower rank L-functions at shifted arguments $L(s,\pi)=\prod_{i=1}^r L(s_i,\pi_i)$, $Z(s,w)$ is a multiple Dirichlet series of the form

$$Z(s_1,s_2,\cdots,s_r,w) \;=\; \sum_{d} \frac{\left(\prod_{i=1}^r L(s_i,\pi_i\times\chi_d)\right)a(\{s_i\},\{\pi_i\},d)}{|d|^w}$$

for suitable weight factors a. One may study these series by similar methods.

2.2. A first example. Why is a series such as (2.1) a reasonable thing to construct? We begin with the case of GL(1). Let $j(\gamma,z)$ be the theta multiplier

$$j(\gamma,z)=\epsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}, \qquad \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(4),$$

where $\epsilon_d=1$ if $d\equiv 1 \bmod 4$, $\epsilon_d=i$ if $d\equiv 3 \bmod 4$, $\left(\frac{c}{d}\right)$ is a (quadratic) Kronecker symbol, and the square root is chosen so that $-\pi/2<\arg((cz+d)^{1/2})\le\pi/2$. Let $E(z,s)$ be the half-integral weight Eisenstein series

$$\widetilde{E}(z,s) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} j(\gamma,z)^{-1}\Im(\gamma z)^s.$$

Maass [**45**] showed in 1937 that the m^{th} Fourier coefficient of $\widetilde{E}(z,s)$ is essentially equal to $L(2s,\chi_m)$ where χ_m is a quadratic character given by a Legendre symbol.

Here *essentially equal* means that this is correct up to Euler 2-factor, archimedean factors (suppressed from the notation) and most importantly correction factors that adjust the formulas when m is not square-free. The correction factor multiplying $L(2s,\chi_m)$ is a product of Dirichlet polynomials in $|v|^{-s}$ at the places v such that $\mathrm{ord}_v(m)\ge 2$.

Given any modular form, its Mellin transform is the Dirichlet series formed by summing its Fourier coefficients. Siegel [**52**] applied a Mellin transform to $\widetilde{E}(z,s)$ and observed that

$$\int_0^\infty \left(\widetilde{E}(iy,s)-\text{const term}\right)y^w d^\times y \;\approx \sum_m \frac{L(2s,\chi_m)}{m^w}.$$

Here the $\approx$ is used to remind the reader that 2-factors, archimedean places and correction factors are being suppressed. There is also an issue of normalizing the integral that we do not discuss in detail. This is the fundamental relation that Goldfeld-Hoffstein exploited in [**34**] to obtain asymptotics for

$$\sum_{0<\pm m<X} L(2s,\chi_m).$$

Later Goldfeld-Hoffstein-Patterson [**35**] used similar Eisenstein series over an imaginary quadratic field together with the Asai integral [**1**] to get similar results for L-functions attached to CM elliptic curves, and then Hoffstein and Rosen [**38**] used the method over the rational function field $\mathbb{F}_q(T)$.

Goldfeld and Hoffstein anticipated the difficulty of generalizing this construction to automorphic L-functions of higher degree. They write [**34**]:

> At present, however, we cannot obtain mean value theorems for quadratic twists of an arbitrary L-function associated to an automorphic form... These appear to be difficult problems and their solution may ultimately involve the analytic number theory of $\mathrm{GL}(n)$.

2.3. Examples of multiple Dirichlet series arising from RankinSelberg integrals. The Mellin transform and Asai integral mentioned above are examples of Rankin-Selberg integrals. In fact there are many other examples of Rankin-Selberg integrals that give rise to multiple Dirichlet series. A number of interesting examples can be understood as follows: in Section 2.2 we saw that the Mellin transform, which gives a standard L-function if applied to a GL(2) form of integral weight, gives a multiple Dirichlet series of the desired type when applied to an Eisenstein series of *half-integral weight*. Note that the integral is no longer an Euler product in that case. In a similar way we can look at other integrals that give Euler products—Rankin-Selberg integrals—when applied to an automorphic form. Replacing the automorphic form by a metaplectic Eisenstein series (like the half-integral weight Eisenstein series $\widetilde{E}$), one can hope that the resulting object is an interesting multiple Dirichlet series. We mention a few cases in which this hope is realized.

2.3.1. *Examples:*

(1) Let π be an automorphic representation of GL(2) over $\mathbb{Q}(i)$. In [**12**] Bump, Friedberg, and Hoffstein use π to construct a metaplectic Eisenstein series $\widetilde{E}_\pi$ on the double cover of GSp_4. Now, an integral transformation due to Novodvorsky [**48**] produces the spin L-function when applied to a non-metaplectic automorphic form on GSp_4. When the same transformation is applied to the metaplectic Eisenstein series $\widetilde{E}_\pi$ a multiple Dirichlet series of type (2.1) is created, with $n = r = 2$. The choice of ground field was for convenience. A cleaner approach was found using Jacobi modular forms and presented in [**13**], over ground field $\mathbb{Q}$. For applications to elliptic curves see [**11**]. Another construction of Friedberg-Hoffstein [**31**] obtains this same multiple Dirichlet series using a Rankin-Selberg convolution of π with a half-integral weight Eisenstein series on GL(2). That paper works over an arbitrary number field.

(2) Let π be a GL(3) automorphic form. Work of Bump, Friedberg, Hoffstein, and Ginzburg (unpublished) obtains the double Dirichlet series of (2.1) as an integral of an Eisenstein series on the double cover of GSp_6, or as an integral of an Eisenstein series on $SO(7)$ (these two groups are linked by the theta correspondence).

(3) Suzuki [**54**] and Banks-Bump-Lieman [**2**], generalizing earlier work of Bump and Hoffstein [**16**], showed that there is a metaplectic Eisenstein series on the n-fold cover of $\mathrm{GL}(n)$ (induced from the theta function on the n-fold cover of $\mathrm{GL}(n-1)$) whose Whittaker coefficients are n-th order twists of a given GL(1) L-series. An

integral transformation yields a sum of twists of GL(1):

$$\sum_d \frac{L(s, \xi\chi_d^{(n)})\, a(s, \xi, d)}{|d|^w},$$

where ξ is on GL(1) and is fixed. One should then be able to control such sums; however, the technical difficulties are substantial, as discussed in paragraph 2.3.2 below. In Farmer, Hoffstein, and Lieman [**27**], mean value results for cubic L-series were obtained by this approach. (This series has been studied by Friedberg, Hoffstein, and Lieman [**32**], using a different method that is explained in Section 4.1 below.)

(4) Similarly, working with n-th order twists, A. Diaconu [**24**] studied

$$\sum \frac{|L(s, \chi_m^{(n)})|^2}{|m|^w}.$$

This can be obtained from a Rankin-Selberg integral convolution of the metaplectic Eisenstein series on the n-fold cover of $GL(n)$ described above. Once again, Diaconu used a different strategy to study this integral, as we shall explain.

2.3.2. *Obstructions.* In the above paragraph, we described a number of multiple Dirichlet series that arose as Rankin-Selberg type integrals. Unfortunately, it turns out to be rather difficult to study the series using such constructions. The following obstructions arise:

(1) Truncation: The integrals involving Eisenstein series need to be truncated or otherwise renormalized in order to converge. This can be handled in principle via the general theory of Arthur. It is, however, complicated to do in the situations above;
(2) Bad finite primes: Bad finite primes are difficult to handle in Rankin-Selberg type integrals, unlike the Langlands-Shahidi method. This is particularly true in the case of integrals involving metaplectic automorphic forms, where the primes dividing n present additional complications;
(3) Archimedean places: Integrals of archimedean Whittaker functions arise in the integrals. But the general theory of such integrals is not fully developed. This is possibly the most serious obstruction to this approach.

Since many properties of L-functions are already known, one might hope that one can write down and study multiple Dirichlet series *without* needing to employ Rankin-Selberg integrals. Remarkably, this is possible in many cases, and it is one main goal of this paper, and succeeding papers, to explain how. However, we note that information obtained from metaplectic Eisenstein series does play a key role in the study of higher twists, as we shall explain also explain in Section 4 below.

3. Quadratic Twists

3.1. A heuristic. In Section 2.3 we have seen that a number of double Dirichlet series arose via Rankin-Selberg integrals. Such series necessarily have continuation coming from the integral. Could this have been predicted without the integral? And what happens if one can not find such an integral?

In 1996, Bump, Friedberg and Hoffstein [**14**] presented a heuristic that explains what to expect in the quadratic twist case. We describe it now. Consider a $GL(r)$

L-function

$$L(s,\pi) \;=\; \sum_n c(m)|m|^{-s}.$$

The family of objects of interest is $L(s, \pi \times \chi_d)$, where χ_d varies over quadratic twists; we write

$$L(s,\pi\times\chi_d) \;=\; \sum_m c(m)\left(\frac{d}{m}\right)|m|^{-s}.$$

Note that $\left(\frac{d}{m}\right)$ is zero if $(d,m) > 1$, so this equation is not exactly correct if d is not square-free, but we will not keep track of this complication at the moment. Set

$$Z(s,w) \;=\; \sum_m \frac{L(s,\pi\times\chi_d)}{|d|^w}. \tag{3.1}$$

In fact, this is not the actual definition of the correct multiple Dirichlet series as we are ignoring weight factors and also not specifying the m that we are summing over. We are now in the land of the heuristic and things will get even looser. If we temporarily pretend that all integers are square-free and relatively prime, then we can expand the L-series in the numerator of $Z(s,w)$ and write (for $\Re(s), \Re(w)$ sufficiently large)

$$Z(s,w) \;=\; \sum_d \sum_m c(m)\left(\frac{d}{m}\right)m^{-s}d^{-w}.$$

In this heuristical universe we may as well assume that reciprocity works perfectly with no bad primes. Assuming this, we can reverse the order of summation, obtaining

$$Z(s,w) = \sum_m c(m)L(w,\chi_m)m^{-s}. \tag{3.2}$$

Note that we started with a sum of $L(s,\pi\times\chi_d)$, that is, a sum of twisted $\mathrm{GL}(r)$ L-functions, in (3.1), and ended with a sum of $L(w,\chi_m)$, that is, a sum of twisted $\mathrm{GL}(1)$ L-functions in (3.2)! That is, our sum of Euler products in s is at the same time a sum of Euler products in w! Again, this is only a heuristic, as it assumes $\left(\frac{m}{d}\right) = \left(\frac{d}{m}\right)$ and all numbers are square-free and relatively prime. However it turns out that reality can be made to fit this heuristic remarkably well.

We will now explore the functional equations of these twisted L-functions. For d square-free there is a functional equation sending

$$L(s,\pi\times\chi_d) \;\to\; |d|^{r(\frac{1}{2}-s)}L(1-s,\widetilde{\pi}\times\chi_d), \tag{3.3}$$

as well as one sending

$$L(w,\chi_m) \;\to\; |m|^{\frac{1}{2}-s}L(1-w,\chi_m). \tag{3.4}$$

Thus $Z(s,w)$ satisfies two types of functional equations:

(1) First we have a functional equation under $s \to 1-s$, obtained from (3.3). Because of the power of $|d|$ that is introduced we have, upon substituting into (3.1), $w \to w + r(s-\frac{1}{2})$. Thus

$$Z(s,w) \;\to\; Z(1-s, w + r(s-1/2)). \tag{3.5}$$

(Strictly speaking we should write the right hand side as $\tilde{Z}(1-s, w+r(s-1/2))$ as π is replaced by its contragredient.)

(2) We also have a functional equation under the transformation $w \to 1 - w$, obtained from (3.4). Applying this to (3.2) yields a transformation

$$Z(s, w) \;\to\; Z(s + w - 1/2, 1 - w). \tag{3.6}$$

Note that each of these functional equations goes hand in hand with an extension of $Z(s, w)$, originally defined by an absolutely convergent series in $\Re(s), \Re(w) > 1$, to a larger region. It is convenient to think of these transformations as operating (repeatedly) on a region of definition to extend the function to a larger region, and we will do so below, but strictly speaking one obtains first the continuation to the larger region (by Phragmen-Lindelöf), and then the functional equation on this larger region.

Writing these functional equations carefully would require writing the archimedean factors and also describing a suitable scattering matrix; for the heuristic this level of detail is not needed.

One can apply the functional equations (3.5) and (3.6) successively. They generate a finite group of functional equations for GL(1), GL(2) and GL(3), i.e for $r = 1, 2, 3$ but an infinite group for GL(4) (in fact an affine Weyl group) and higher. This suggests that it should be possible to define a precise, non-heuristic, $Z(s, w)$ that continues to $\mathbb{C}^2$ for GL(1), GL(2) and GL(3) but that significant obstructions may appear for GL(4) and higher.

To go farther, let us consider poles. We expect that there is a pole at $w = 1$, since $\zeta(w)$ arises in equation (3.2) when $d = 1$. This polar line is reflected by the functional equations into a collection of polar lines that will be finite if $r = 1, 2, 3$ and infinite if $r \geq 4$ (see [**14**]). For any fixed s_0 the possibility of a pole at $s = s_0, w = 1$ can be investigated by computing the sum of the contributions from the polar lines that intersect $(s_0, 1)$. If $Z(s, w)$ does in fact have a pole at $(s_0, 1)$, then, by (3.1), this implies the non-vanishing of $L(s_0, \pi \times \chi_d)$ for infinitely many χ_d. Similarly if one can continue to $(s, w) = (1/2, 1)$ and if all epsilon factors at $1/2$ are -1 then one can differentiate with respect to s and set $s = 1/2$. There should still be a pole at $w = 1$ provided that the different polar divisors do not cancel when $s = 1/2$. In that case, one may then obtain a non-vanishing theorem for $L'(1/2, \pi \times \chi_d)$ from the pole of $\frac{\partial}{\partial s} Z(s, w)$ at $s = 1/2, w = 1$. Standard methods involving contour integrals can also give mean value theorems.

In the case of GL(4) and higher the group of functional equations is infinite. If we take this infinite group and use it to translate the line $w = 1$, the poles accumulate and create a barrier to continuation. See [**14**], Section 4, for some elaboration of this point. Because of this we do not expect continuation to all of $\mathbb{C}^2$ when $r \geq 4$. However, if we could get continuation up to the conjectured barrier, that would be very significant; we would get a tremendous amount of information (Lindelöf in twisted aspect, simultaneous non-vanishing at the center of the critical strip). At the moment this problem seems extremely challenging.

The situation for GL(1), GL(2) and GL(3) is different. There *we can make the heuristic rigorous* and thereby prove continuation to $\mathbb{C}^2$ without using Rankin-Selberg integrals! Applications (non-vanishing, mean-value theorems) then follow. The key point is to take advantage of the finite group of functional equations, and Hartogs' Continuation Principle.

3.2. Hartogs' continuation principle and Bochner's tube theorem. To overcome the obstructions that arise in the Rankin-Selberg integral method of studying multiple Dirichlet series, we shall employ Hartogs' Principle in a stronger form due to Bochner. Let us describe this now. We recall the definition:

Definition 3.1 (Tube Domain). An open set $\Omega \subset \mathbb{C}^m$ is called a *tube domain* if there is an open set $\omega \in \mathbb{R}^m$ such that $\Omega = \{s \in \mathbb{C}^m : \Re(s) \in w\}$. We write $\Omega = T(\omega)$ to denote this relation.

If $R \subset \mathbb{R}^m$ or $\mathbb{C}^m$ and $m \geq 2$, let $\hat{R}$ be the *convex hull* of R. It is easy to see that if $\Omega = T(\omega)$ then $\hat{\Omega} = T(\hat{\omega})$. Then the relevant result is

Theorem 3.2. *(see Hörmander* [**39**]*, Theorem 2.5.10) If Ω is a connected tube domain, then any holomorphic function in Ω can be extended to a holomorphic function on $\hat{\Omega}$.*

Thus if we can continue a meromorphic function whose polar divisor is a finite number of hyperplanes to Ω it automatically extends meromorphically to $\hat{\Omega}$, since its product with a finite number of linear factors is holomorphic. In many cases this is exactly what occurs with multiple Dirichlet series.

The theorem above is due to Bochner. A weaker result of Hartogs states that there are no compact holes in domains of holomorphy in more than one complex variable.

3.3. Sketch of the continuation of $Z(s,w)$ to $\mathbb{C}^2$ for $\mathrm{GL}(r)$ for $r \leq 3$. We can now sketch the continuation of $Z(s,w)$ for π on $\mathrm{GL}(r)$ with $r \leq 3$. First, suppose that we can introduce some weight functions $a(s,\pi,d)$ so that the interchange of summation is actually valid. The original heuristic interchange of summation implicitly assumed everything was square-free, which is not the case. We assume now that with appropriate weight factors this interchange will in fact work. The weight factors do exist; see Sections 3.4, 3.5 below for more details. Moreover, as we shall explain there, they are unique—for $r \leq 3$ there are unique factors that allow the sum of Euler products in s to equal a sum of Euler products in w [**15**]!

The relevant series to look at is

$$Z(s,w) = \sum L(s, \pi \times \chi_d)\, a(s,\pi,d)\, \xi(d)|d|^{-w}, \tag{3.7}$$

where ξ is on $\mathrm{GL}(1)$ and π is an automorphic form on $\mathrm{GL}(r)$ with $r \leq 3$. When the weight factors $a(s,\pi,d), b(w,\xi,\pi,m)$ are chosen correctly, this can be rewritten after applying quadratic reciprocity as

$$Z(s,w) = \sum L(w, \xi\chi_m)\, b(w,\xi,\pi,m)\, |m|^{-s}. \tag{3.8}$$

In addition to allowing this interchange of summation, the weighting factors, when multiplied by the L-functions, satisfy the functional equations

$$L(s,\pi \times \chi_d)a(s,\pi,d) \;\to\; |d|^{r(\frac{1}{2}-s)}L(1-s,\widetilde{\pi} \times \chi_d)a(1-s,\widetilde{\pi},d), \tag{3.9}$$

and

$$L(w,\xi\chi_m)b(w,\xi,\pi,m) \;\to\; |m|^{\frac{1}{2}-s}L(1-w,\bar{\xi}\chi_m)b(1-w,\bar{\xi}\widetilde{\pi},m). \tag{3.10}$$

The existence of these weighting factors for $r = 1, 2$ and the bounds

$$|a(s,\pi,d)| \ll_\epsilon |d|^\epsilon \text{ and } |b(w,\xi,\widetilde{\pi},m)| \ll_\epsilon |m|^{\frac{1}{2}+\epsilon} \text{ for } \Re(s), \Re(w) > 3/2 \tag{3.11}$$

will be shown in the following section.

From (3.9),(3.10),(3.11) and the Phragmen-Lindelöf principle, we deduce the convexity bounds

$$(1-s)^{\ell} L(s, \pi \times \chi_d) a(s, \pi, d) \ll_\epsilon |d|^3 \text{ for } \Re(s) > -\tfrac{1}{2} \tag{3.12}$$

and

$$(1-w)^k L(w, \xi\chi_m) b(w, \xi, \pi, m) \ll_\epsilon |m| \text{ for } \Re(s) > -\tfrac{1}{2} \tag{3.13}$$

where l is the order of the pole of $L(s, \pi \times \chi_d)$ at $s = 1$ and k is the order of the pole of $L(w, \xi\chi_m)$ at $w = 1$. (Such poles occur only if π is non-cuspidal with central character χ_d or if $\xi = \chi_m$.) Thus by absolute convergence, the representation (3.7) of $Z(s, w)$ defines an analytic function for $\Re(s) > -\frac{1}{2}, \Re(w) > 4$ and the representation (3.8) is analytic for $\Re(w) > -\frac{1}{2}, \Re(s) > 2$. Let X be the union of these two regions. Then X is a connected tube domain. Let G be the finite group of transformations of $\mathbb{C}^2$ generated by

$$(s, w) \mapsto (1-s, w + r(s - \tfrac{1}{2})) \text{ and } (s, w) \mapsto (s + w - \tfrac{1}{2}, 1 - w) \tag{3.14}$$

As indicated in Section 3.1, the double Dirichlet series $Z(s, w)$ has an invariance with respect to this group G. Moreover, the tube domain X contains the complement of a compact subset of a fundamental domain for the action of G on $\mathbb{C}^2$. Therefore the union of the translates of X by G is Ω, say, a connected tube domain which is the complement of a compact subset of $\mathbb{C}^2$. It follows that we can analytically continue $Z(s, w)$ to the set Ω, and in fact, $P(s, w)Z(s, w)$ is holomorphic on Ω, where $P(s, w)$ is a finite product of linear terms which clear the translates of the possible polar lines $s = 1, w = 1$ of $Z(s, w)$. We now apply Theorem 3.2 to analytically continue $Z(s, w)$ to $\mathbb{C}^2$. A similar argument is presented elsewhere in this volume in [**7**], Section 1, and the reader may wish to see the figure illustrating it there.

For example, let π be an automorphic representation of GL(3) with trivial central character. The group G is dihedral of order 12. In [**15**] it is shown that

$$w(w-1)(3s + w - 5/2)(3s + 2w - 3)(3s + w - 3/2) \times \text{bad prime factor} \times Z(s, w)$$

has an analytic continuation to $\mathbb{C}^2$.

Similarly, the multiple Dirichlet series (with suitable weight factors) corresponding to GL(1) × GL(1) and GL(1) × GL(2), resp. GL(1) × GL(1) × GL(1), given by (2.1) meromorphically continue to $\mathbb{C}^3$ resp. $\mathbb{C}^4$ with a finite number of polar hyperplanes. The weight factors needed to make the heuristic rigorous (i.e. to show that a sum of Euler products in the s_i is also a sum of Euler products in w) are once again unique.

Though the heuristics are easiest to explain over $\mathbb{Q}$, we emphasize that the method works over a general global field [**29**],[**30**]. To do so, one must pass to a ring of S-integers that has class number one, and look at a finite dimensional vector space of multiple Dirichlet series. This space is stable under the functional equations, and the method applies. An additional complication is the epsilon-factors that arise in the functional equations for automorphic L-functions. As shown in Fisher and Friedberg [**29, 30**], it is possible to sieve the d's using a finite set of characters so that for d, d' in the same class,

$$\epsilon(1/2, \pi \times \chi_d) = \epsilon(1/2, \pi \times \chi_{d'}).$$

This is crucial, and allows one to apply the functional equation to the sum of L-functions $Z(s, w)$ and obtain an object that is a finite linear combination of similar double Dirichlet series, rather than series with new weights coming from the epsilon factors.

Since the base field may be general, one may study the functions $Z(s,w)$ for function fields. In that case, for π on $GL(r)$ with $r \leq 3$, $Z(s,w)$ reduces to a rational function in q^{-s} and q^{-w} (where q is the cardinality of the field of constants) with a specified denominator; this comes from the functional equations. For example, given any algebraic curve over a finite field one gets a finite dimensional vector space of rational functions of two complex variables; see [**29**] for details and examples, and Section 5 below for a discussion of the rational function field case. It would be intriguing to give a cohomological interpretation of these rational functions, but so far no one has done so.

In the next two sections, we discuss the crucial ingredient in making the heuristic rigorous—the interchange of summation—in more detail. Then in Sections 3.6, 3.7 we describe several applications of the method.

3.4. The interchange of summation: GL(1) **and** GL(2) **cases.** In this section, we explain the interchange of summation that relates (3.7) and (3.8) when π is on GL(1) or GL(2) in more detail. For the moment, we simply exhibit the weight factors $a(s,\pi,d)$, $b(w,\xi,\pi,m)$ directly. One might ask what conditions these weight factors must satisfy if the method is to work, whether or not they are unique (they are), and how they can be determined. These questions are taken up for π on GL(2) in the following section; the case of GL(1) is similar. The weight factors and the interchange for π on GL(3), as well as the uniqueness of these weight factors, is more complicated, and we refer the reader to [**15**] for details. (For GL(4) and beyond the interchange, functional equation and Euler product properties are not enough to force uniqueness; see [**15**].)

Throughout this section we will write sums without specifying the precise set we are summing over. For convenience, the reader may imagine that we are summing over positive rational integers prime to the conductor. Over a general number or function field, one sums over a suitable set of ideals prime to a finite set S, and adjusts the definitions to be independent of units. We refer to [**29**], Section 1, or to Brubaker and Bump [**5**] for details.

3.4.1. *Sums of* GL(1) *quadratic twists.* Let π be an idèle class character. Let $d = d_0 d_1^2$ where d_0 is square-free. We write $\chi_d = \chi_{d_0}$ for the character given by the quadratic Kronecker symbol $\chi_d(a) = (\frac{a}{d_0})$ if $(a, d_0) = 1$, and extend this function to take value 0 if $(a, d_0) > 1$. Let $a(s,\pi,d)$ be given by

$$a(s,\pi,d) = \sum_{e_1 e_2 | d_1} \mu(e_1)\, \chi_d(e_1)\, \pi(e_1 e_2^2) |e_1|^{-s} |e_2|^{1-2s}. \tag{3.15}$$

Here $\mu(e_1)$ is a Möbius function. (This factor arises in the Fourier expansion of the half-integral Eisenstein series $\tilde{E}(z, s/2)$ described in Section 2.2 above; see [**37**].) Note that the estimate (3.11) holds for $a(s,\pi,d)$. Then we have

$$\begin{aligned} a(s,\pi,d) &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\, \chi_d(e_1)\, \pi(e_1 e_2^2)\, |e_1|^{-s} |e_2|^{1-2s} \\ &= \pi(d_1^2) |d_1|^{1-2s} \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\, \chi_d(e_1)\, \pi^{-1}(e_1 e_3^2)\, |e_1|^{s-1} |e_3|^{2s-1}. \end{aligned} \tag{3.16}$$

Thus $a(s,\pi,d)$ satisfies the functional equation

$$a(s,\pi,d) = \pi(d_1^2)|d_1|^{1-2s} a(1-s,\pi^{-1},d). \tag{3.17}$$

Since the conductor of χ_d is d_0 (remember, we will ultimately avoid even places by passing to a ring of S-integers), $L(s,\pi\times\chi_d)$ is equal to a factor involving the bad places times $\epsilon(\pi\chi_d)|d_0|^{1/2-s}L(1-s,\pi^{-1}\times\chi_d)$, where $\epsilon(\pi\chi_d)$ is the central value of a global epsilon-factor.

Recall that $Z(s,w)$ (or $Z(s,w;\pi,\xi)$ to be more precise) is given by

$$Z(s,w;\pi,\xi) = \sum_d L(s,\pi\times\chi_d)a(s,\pi,d)\xi(d)|d|^{-w}.$$

Here ξ is a second idèle class character. Substituting in the functional equations for $L(s,\pi\times\chi_d)$ and for $a(s,\pi,d)$, one obtains a functional equation relating $Z(s,w;\pi,\xi)$ to $Z(1-s,w+s-1/2,\tilde{\pi},\xi)$ (cf. (3.9)). Notice that a factor of $|d_0|^{1/2-s}$ comes from the functional equation for the GL(1) L-function, arising since the conductor changes by d_0 upon twisting. This factor fits exactly with the $|d_1|^{1-2s}$ arising from the functional equation (3.17) of the weight factor $a(s,\pi,d)$, and it is this combination that shifts w to $w+s-1/2$. We also have that $\epsilon(\pi\chi_d)\pi(d_1^2)$ is essentially constant—this is true for d congruent to 1 modulo a sufficiently large ideal, and so the epsilon factors do not create a series of a fundamentally different type after sending $s\mapsto 1-s$. See [**29**], Corollary 2.3, for more about the epsilon factors ([**29**] works over a function field but the result is similar over a number field) and [**29**], Theorem 2.6, for the exact functional equation.

We turn to the rewriting of $Z(s,w)$ as a sum of Euler products in w, which leads to the second functional equation (3.10). We always work in the domain in which these sums converge absolutely ($\Re(s),\Re(w)>1$ will do). Substituting in the definition of $a(s,\pi,d)$ and expanding the L-function $L(s,\pi\times\chi_d)$ as a sum, we obtain

$$\begin{aligned} Z(s,w;\pi,\xi) = \sum_{d=d_0d_1^2}\sum_m\sum_{e_1e_2|d_1} &\xi(d)\,\pi(m)\,\chi_d(m)\,\mu(e_1)\,\chi_d(e_1)\,\pi(e_1e_2^2)\\ &\times|m|^{-s}|d|^{-w}|e_1|^{-s}|e_2|^{1-2s}.\end{aligned}$$

The quadratic symbols give 0 unless $(d_0,me_1)=1$. Replace m by $m'=me_1$. The sum over m and e_1 becomes $\sum_{m',e_1}\pi(m')\,\chi_d(m')|m'|^{-s}\mu(e_1)$, where in the sum $e_1|m'$, $e_1|(d_1/e_2)$. The sum over the Möbius function vanishes unless $(m',d_1/e_2)=1$, in which case it is 1. So we obtain

$$\sum_{d=d_0d_1^2}\sum_{e_2|d_1}\sum_{\substack{m'\\(m',d_0d_1/e_2)=1}}\xi(d)\,\pi(e_2^2)\,\pi(m')\,\chi_d(m')\,|d|^{-w}|e_2|^{1-2s}|m'|^{-s},$$

Now replace d by de_2^2. This gives a sum over d,m',e_2 subject to the constraint $(d,m')=1$. The sum over e_2 gives $L(2s+2w-1,\pi^2\xi^2)$. Thus we obtain the equation (dropping the prime on the variable m')

$$Z(s,w;\pi,\xi) = L(2s+2w-1,\pi^2\xi^2)\sum_{(d,m)=1}\xi(d)\,\pi(m)\,\chi_d(m)\,|d|^{-w}|m|^{-s}. \tag{3.18}$$

Modulo dealing carefully with quadratic reciprocity, we see that we have a functional equation $Z(s,w;\pi,\rho)=Z(w,s;\rho,\pi)$. (For the precise statement, see [**29**], Theorem 3.3.) This gives (3.8) and the second desired functional equation (3.10), and allows us to establish the continuation of $Z(s,w)$ to $\mathbb{C}^2$.

We remark that a similar proof applies to n-fold twists, provided that one writes $d = d_0 d_1^n$ with d_1 n-th power free and one uses the weight function

$$a(s,\pi,d) = \sum_{e_1 e_2 | d_1} \mu(e_1)\, \chi_d(e_1)\, \pi(e_1 e_2^n)\, |e_1|^{-s} |e_2|^{n-1-ns}.$$

See [**32**], Proposition 2.1, as well as Section 4.1 below.

3.4.2. *Sums of* GL(2) *quadratic twists.* In this section we follow the approach of Fisher and Friedberg [**30**] to present the GL(2) computation. Suppose now that π is cuspidal on GL(2) with $L(s,\pi) = \prod_v ((1-\pi_1(v)|v|^{-s})(1-\pi_2(v)|v|^{-s}))^{-1}$. Here $\pi_1(v)$, $\pi_2(v)$ are the Satake parameters for π_v. (Once again we are really taking the L-function with the primes in a finite set S of bad places removed, but we omit this from the notation.) Extend π_1, π_2 multiplicatively to be functions defined on ideals prime to S. Let

$$A(s,\pi,d) = a(s,\pi_1,d)\, a(s,\pi_2,d) \tag{3.19}$$

where the factors on the right hand side are given by (3.15). It will turn out that $A(s,\pi,d)$ is closely related to the desired GL(2) weight function $a(s,\pi,d)$; see (3.23) below. For ξ on GL(1), we set

$$Z_A(s,w;\pi,\xi) = \sum L(s,\pi\times\chi_d)\, A(s,\pi,d)\, \xi(d)\, |d|^{-w}.$$

From the functional equation (3.17) for the GL(1) weight function, we obtain

$$A(s,\pi,d) = \chi_\pi(d_1^2)|d_1|^{2-4s} A(1-s,\tilde{\pi},d), \tag{3.20}$$

where as above $d = d_0 d_1^2$ with d_0 square-free, and where χ_π is the central character of π. From this and the functional equation for the L-function $L(s,\pi\times\chi_d)$, one immediately obtains a functional equation for $Z_A(s,w)$ with respect to the transformation $(s,w)\mapsto(1-s, w+2s-1)$.

A second functional equation is obtained by proving an analogue of (3.18). Namely, we have the key (and remarkable) formula

$$\begin{aligned} L(2s+2w-1,\chi_\pi\xi^2)\, Z_A(s,w;\pi,\xi) &= L(4s+2w-2,\chi_\pi^2\xi^2) \\ &\times \sum_{m_1,m_2} \pi_1(m_1)\,\pi_2(m_2)\, L(w,\xi\chi_{m_1m_2})\, a(w,\xi,m_1m_2)\, |m_1m_2|^{-s}. \end{aligned} \tag{3.21}$$

Here $a(w,\xi,m_1m_2)$ is the GL(1) weight factor, given by (3.15). Though the full details are too long to include here (see [**30**], Section 2), we will present a sketch of the proof of this result.

First, substituting in the Dirichlet series for $L(s,\pi\times\chi_d)$ and changing variables to sum the two Möbius functions, we find that

$$Z_A(s,w;\pi,\xi) = \sum_{m_1,m_2,d,e_1,e_2} \pi_1(m_1)\,\pi_2(m_2)\,\xi(d)\,\chi_d(m_1e_1^{-2})\,\chi_d(m_2e_2^{-2})\,|e_1e_2|\,|m_1m_2|^{-s}|d|^{-w}$$

where the summation variables are subject to the restrictions $(m_i,d)=e_i^2$, $i=1,2$ ([**30**]), Proposition 2.2). Introducing a variable $e=(e_1,e_2)$, one can rewrite the sum and pull out an L-function $L(4s+2w-2,\chi_\pi^2\xi^2)$. Then replacing d by $de_1^2e_2^2$ one arrives at a sum over variables m_1,m_2,d,e_1,e_2 subject to the constraints $e_i^2|m_i$

$(i = 1, 2)$, $(e_1, m_2) = (e_2, m_1) = 1$, and $(d, m_1 m_2 e_1^{-2} e_2^{-2}) = 1$. Replacing this last equation in the standard way by a sum of Möbius functions, one can once again obtain an L-function $L(w, \xi\chi_{m_1 m_2})$. Then multiplying by $L(2s + 2w - 1, \chi_\pi \xi^2)$, writing this last as a sum (over g) and changing several summation variables, we obtain

$$\begin{aligned}(3.22)\quad L(2s+2w-1,\chi_\pi\xi^2)\, Z_A(s,w;\pi,\xi) &= L(4s+2w-2,\chi_\pi^2\xi^2)\times \\ \sum_{m_1.m_2,d,e_1,e_2,g} \pi_1(m_1)\,\pi_2(m_2)\, L(w,\xi\chi_{m_1m_2})\,\mu(d)\,\chi_{m_1m_2}(d)\,\xi(de_1^2e_2^2g^2) \\ |m_1m_2|^{-s}|d|^{-w}|e_1e_2g|^{1-2w}\end{aligned}$$

with summation conditions $ge_i^2|m_i$ $(i = 1, 2)$, $(de_1e_2g)^2|m_1m_2$, $(e_1, m_2g^{-1}) = (e_2, m_1g^{-1}) = (d, (m_1m_2)') = 1$, where the prime denotes the square-free part. But given m_1, m_2, there is a one-to-one correspondence between triples (e_1, e_2, g) such that $ge_i^2|m_i$ $(i = 1, 2)$, $(e_1, m_2g^{-1}) = (e_2, m_1g^{-1}) = 1$ and numbers f such that $f^2|m_1m_2$; the correspondence takes (e_1e_2, g) to $f = e_1e_2g$ (see [**30**], Lemma 2.5). Applying this, equation (3.22) can be rewritten

$$\begin{aligned}L(2s+2w-1,\chi_\pi\xi^2)\, Z_A(s,w;\pi,\xi) &= L(4s+2w-2,\chi_\pi^2\xi^2)\times \\ \sum_{m_1.m_2,d,f} \pi_1(m_1)\,\pi_2(m_2)\, L(w,\xi\chi_{m_1m_2})\,\mu(d)\,\chi_{m_1m_2}(d)\,\xi(df^2) \\ |m_1m_2|^{-s}|d|^{-w}|f|^{1-2w}\end{aligned}$$

where in the sum $d^2f^2|m_1m_2$, $(d, (m_1m_2)') = 1$. The sum over d and f gives the GL(1) weight factor $a(w, \xi, m_1m_2)$, and equation (3.21) follows.

Finally, let us give the GL(2) weight factors and explain the relation between formula (3.21) and the equality of (3.7) and (3.8) for suitable weight factors. The GL(2) weight factor is given by:

$$a(s,\pi,d) = \sum_{e^2|d} |e|^{1-2s}\chi_\pi(e)\, A(s,\pi,de^{-2}). \tag{3.23}$$

Since the quantity $|e|^{1-2s}\, A(s, \pi, de^{-2})$ satisfies precisely the same functional equation (3.20) as $A(s, \pi, d)$ itself, we see that $Z(s, w; \pi, \xi)$ satisfies a functional equation with respect to the transformation $(s, w) \mapsto (1 - s, w + 2s - 1)$. As for the equality of (3.7) and (3.8) (for suitable b), substituting (3.19), (3.23) in to (3.7), and interchanging summation one obtains

$$Z(s,w;\pi,\xi) = \sum_{d,e} L(s,\pi\times\chi_d)\, a(s,\pi_1,d)\, a(s,\pi_2,d)\, \xi(de^2)\, \chi_\pi(e)\, |e|^{1-2s-2w}\, |d|^{-w}.$$

Summing over e, we see that

$$\begin{aligned}Z(s,w;\pi,\xi) &= L(2s+2w-1,\chi_\pi\xi^2)\sum_d L(s,\pi\times\chi_d)\, a(s,\pi_1,d)\, a(s,\pi_2,d)\xi(d)\, |d|^{-w} \\ &= L(2s+2w-1,\chi_\pi\xi^2)\, Z_A(s,w;\pi,\xi).\end{aligned}$$

We may hence apply equation (3.21) in order to see that $Z(s, w; \pi, \xi)$ is equal to a sum of GL(1) L-functions in w, as desired.

3.5. More on the interchange of summation: an example of the uniqueness principle. The interchanges of summation exhibited in the previous section raise the following questions: (a) are the weight factors given there canonical? and (b) how can one find such factors, if one does not know them in advance? In this section we answer these questions when π is on GL(2). We will explain how to determine the weight factors of the multiple Dirichlet series directly, thereby establishing a uniqueness principle. More precisely, we will suppose that the weight factor has three properties: (i) it has an Euler product; (ii) it gives the proper functional equation for the product $L(s, \pi \times \chi_d)\, a(s, \pi, d)$ even when d is not square-free; and (iii) it has the correct properties with respect to interchange of summation. Under these assumptions, we will show that the weight factor for generic primes is unique, and in fact may be determined completely. (We will still ignore bad primes, for convenience.) The approach given here works for GL(1) (an easy exercise), and it also generalizes to other situations, such as GL(3) ([**15**]), where the weight factors are too complicated to guess.

So suppose that π is an automorphic representation of GL(2), with standard L-function

$$L(s, \pi) \;=\; \sum \frac{c(m)}{m^s}.$$

For convenience we take the central character of π to be trivial.

Write $d = d_0 d_1^2$ with d_0 square-free. We begin by assuming that

$$a(s, \pi, d) = P(s, d_0 d_1^2),$$

where $P(s, d_0 d_1^2)$ is a Dirichlet polynomial, that is a polynomial in m^{-s} for a finite number of m (the factors $P(s, d_0 d_1^2)$ depend on π, but we suppress this from the notation).

What properties should $P(s, d_0 d_1^2)$ have? For the functional equation to work out correctly we require

$$P(s, d_0 d_1^2) \;=\; d_1^{2-4s} P(1-s, d_0 d_1^2). \tag{3.24}$$

We also require that there be an Euler product expansion for P, namely

$$\begin{aligned} &P(s, d_0 d_1^2) \\ &\quad= \prod_{p^\alpha || d_1} \left(1 + a(d_0 p^{2\alpha}, 1) p^{-s} + a(d_0 p^{2\alpha}, 2) p^{-2s} + \cdots + a(d_0 p^{2\alpha}, 4\alpha) p^{-4\alpha s}\right), \end{aligned} \tag{3.25}$$

where the a's are coefficients to be determined. Note that each factor is forced to end at $p^{-4\alpha s}$ by (3.24). In fact (3.24) implies the recursion relation

$$a(d_0 p^{2\alpha}, k) = p^{k-2\alpha} a(d_0 p^{2\alpha}, 4\alpha - k)$$

for $0 \le k \le 4\alpha$.

For an interchange in the order of summation to work nicely one would like to have the following hold:

$$\sum \frac{L(s, \pi \times \chi_{d_0}) P(s, d_0 d_1^2)}{(d_0 d_1^2)^w} = \sum \frac{L(w, \chi_{m_0}) Q(w, m_0 m_1^2)}{(m_0 m_1^2)^s}. \tag{3.26}$$

Here the $Q(w, m_0 m_1^2)$ should be Dirichlet polynomials with Euler products similar to P. In fact, for the functional equations to work out properly we should have

$$Q(w, m_0 m_1^2) = m_1^{1-2w} Q(1-w, m_0 m_1^2) \tag{3.27}$$

with

$$Q(w, m_0m_1^2) = \prod_{p^\beta || m_1} \big(b(m_0p^{2\beta}, 0) + b(m_0p^{2\beta}, 1)p^{-w} + b(m_0p^{2\beta}, 2)p^{-2w} + \cdots + b(m_0p^{2\beta}, 4\beta)p^{-4\beta w}\big)$$

and the recursion relation

$$b(m_0p^{2\beta}, k) = p^{k-\beta} b(m_0p^{2\beta}, 2\beta - k), \tag{3.28}$$

holding for $0 \le k \le 2\beta$. Notice that we can allow the first term of the Euler product to equal 1 on one side of the equation, but we do not have that freedom on the other.

Let us now consider the coefficients of 1^{-s} on both sides of (3.26). This is easily done by letting $s \to \infty$. As the coefficients must be equal, (3.26) implies that

$$\sum \frac{1}{(d_0d_1^2)^w} = \zeta(w)Q(w, 1),$$

i.e that $Q(w, 1) = 1$. Similarly, letting $w \to \infty$ and equating the coefficients of 1^{-w} we see that

$$L(s, \pi) = \sum \frac{b(m_0m_1^2, 0)}{(m_0m_1^2)^s},$$

Implying that

$$b(m_0m_1^2, 0) = c(m_0m_1^2)$$

for all $m = m_0m_1^2$.

We continue now, equating coefficients of p^{-s} on both sides of (3.26). For fixed square-free d_0 this yields the relation

$$\sum_{d_1} \frac{\chi_{d_0}(p)c(p)}{(d_0d_1^2)^w} + \sum_{p|d_1} \frac{a(d_0d_1^2, 1)}{(d_0d_1^2)^w} = L(w, \chi_p) = \sum \frac{\chi_p(d_0d_1^2)}{(d_0d_1^2)^w} = \sum_{(p,d_1)=1} \frac{\chi_{d_0}(p)}{(d_0d_1^2)^w}.$$

As a consequence of ignoring bad primes we are assuming that reciprocity is perfect ($\chi_{d_0}(p) = \chi_p(d_0)$). It now follows immediately that

$$a(d_0d_1^2, 1) = -\chi_{d_0}(p)c(p)$$

for all $p|d_1$.

Evaluating the coefficient of p^{-w} on each side of (3.26) yields, for fixed square-free m_0,

$$L(s, \pi \times \chi_p) = \sum_{m_1} \frac{\chi_{m_0}(p)c(m_0m_1^2)}{(m_0m_1^2)^s} + \sum_{p|m_1} \frac{b(m_0m_1^2, 1)}{(m_0m_1^2)^s}.$$

As

$$L(s, \pi \times \chi_p) = \sum \frac{\chi_p(m_0m_1^2)c(m_0m_1^2)}{(m_0m_1^2)^s}$$

it thus follows that

$$b(m_0m_1^2, 1) = -c(m_0m_1^2)\chi_{m_0}(p)$$

for all $p|m_1$. Referring to the recursion relation (3.28) and combining this with the above we see that in the case $\beta = 1$ we have now determined the first Q polynomial:

$$Q(w, p^2) = c(p^2)(1 - p^{-w} + p^{1-2w}).$$

Computing the coefficient of p^{-2s} one obtains from the left hand side of (3.26)

$$\sum_{d_1} \frac{\chi_{d_0}(p^2)c(p^2)}{(d_0 d_1^2)^w} + \sum_{p|d_1} \frac{\chi_{d_0}(p)c(p)a(d_0 d_1^2, 1)}{(d_0 d_1^2)^w} + \sum_{p^2|d_1} \frac{a(d_0 d_1^2, 2)}{(d_0 d_1^2)^w}.$$

Combining this with the Hecke relation $c(p)^2 = c(p^2) + 1$ and the information $a(d_0 d_1^2, 1) = -\chi_{d_0}(p)c(p)$ obtained above this reduces to

$$\sum_{(p, d_0 d_1^2)=1} \frac{1}{(d_0 d_1^2)^w} + \sum_{p^2|d_1} \frac{a(d_0 d_1^2, 2)}{(d_0 d_1^2)^w}.$$

The right hand side of (3.26) is

$$\zeta(w)Q(w, p^2) = c(p^2)\left(\sum_{(p,d)=1} \frac{1}{d^w} + p\sum_{p|d_1} \frac{1}{(d_0 d_1^2)^w}\right).$$

Equating the above two expressions we obtain

$$a(d_0 d_1^2) = 1$$

if $p||d_1$ and

$$a(d_0 d_1^2) = 1 + pc(p^2)$$

if $p^2|d_1$. Thus because of the recursion relations we have completely determined the first P polynomial:

$$P(s, d_0 p^2) = 1 - \chi_{d_0}(p)c(p)p^{-s} + p^{-2s} - p\chi_{d_0}(p)c(p)p^{-3s} + p^{2-4s}.$$

This process can be continued, leading to a complete evaluation of the P and Q polynomials.

3.6. An application of the continuation of $Z(s, w)$: quadratic twists of GL(3). In this section we describe the consequences of the continuation to $\mathbb{C}^2$ of the multiple Dirichlet series $Z(s, w)$ in more detail when π is on GL(3). Recall that if π' is a cuspidal automorphic representation of GL(2) then the Gelbart-Jacquet lift $\mathrm{Ad}^2(\pi')$ is an automorphic representation of GL(3) [**33**]. At good places v this map is specified by the behavior of the local L-functions: if

$$L(s, \pi'_v) = \left((1 - \alpha_v|v|^{-s})(1 - \beta_v|v|^{-s})\right)^{-1}$$

then

$$L(s, \mathrm{Ad}^2(\pi'_v)) = \left((1 - \alpha_v\beta_v^{-1}|v|^{-s})(1 - |v|^{-s})(1 - \alpha_v^{-1}\beta_v|v|^{-s})\right)^{-1}.$$

(If π' is self adjoint this is the symmetric square lift.) In [**15**] the following is proved:

THEOREM 3.3. *Let π' be on $GL_2(\mathbb{A}_\mathbb{Q})$. Let M be a finite set of places including 2, ∞, primes dividing the conductor of π'. Then there exist infinitely many quadratic characters χ_d such that d falls in a given quadratic residue class mod v for all $v \in M$ (mod 8 if $v = 2$) and such that $L(\frac{1}{2}, \mathrm{Ad}^2(\pi') \times \chi_d) \neq 0$.*

In this result, the ground field is chosen to be $\mathbb{Q}$ solely for convenience; the method works in general. Moreover, with a little more work one could specify χ_v for all places $v \in M$. One should also be able to establish a similar result for GL(3) automorphic representations that are not lifts from GL(2) by a similar method. Theorem 3.3 is proved by continuing a suitable double Dirichlet series.

Applying Tauberian techniques to the previous theorem one gets

THEOREM 3.4. *Suppose π is automorphic on $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ with trivial central character. Then for $\sigma = \pm 1$ we have*

$$\sum_{d>0} L_M(\frac{1}{2}, \pi, \chi_{\sigma d}) a(\frac{1}{2}, \pi, \sigma d) e^{-d/X} \;=\; CX \log X + C'X + C'' + O(X^{3/4}),$$

where C is a non-zero multiple of

$$\lim_{s \to 1/2} (s - 1/2) L_M(2s, \pi, \mathrm{sym}^2).$$

The term C arises by contour integration as the leading coefficient of the second order pole at $w = 1$. Note that by equation (3.8), this residue arises from the summands indexed by m a perfect square, when ξ is trivial, so it is approximately $\sum c(m^2)|m|^{-2s}$, which is related to $L(2s, \pi, \mathrm{sym}^2)$.

To complete the proof of Theorem 3.3, suppose that $\pi = \mathrm{Ad}^2(\pi')$. Then

$$L(s, \pi, \mathrm{sym}^2) \;=\; \zeta(s) L(s, \mathrm{sym}^4(\pi'), \chi^2_{\pi'}). \tag{3.29}$$

Here $\chi_{\pi'}$ denotes the central character of π'. Using this equality, one can see that $L(s, \pi, \mathrm{sym}^2)$ has a simple pole at $s = 1$. The proof in [**15**] uses the Kim-Shahidi result on the automorphicity of $\mathrm{sym}^4(\pi')$ as well as the Jacquet-Shalika nonvanishing theorem to conclude that the second term does not vanish at $s = 1$, and hence that $C \neq 0$. Prof. Shahidi has kindly informed us that a simpler proof that $L(1, \mathrm{sym}^4(\pi'), \chi^2_{\pi'}) \neq 0$ is available in an older paper of his.

If we take an automorphic representation on GL(3) that is not a lift then $C = 0$. Surprisingly, this thus gives an *analytic way to tell if an automorphic representation on* GL(3) *is or is not a lift from* GL(2)*:* the cases are separated by the asymptotic behavior of their quadratically-twisted L-functions.

Returning to general π on GL(3), and looking at the residue of the series $Z(s, w)$ at $w = 1$, one obtains a proof that for any π on GL(3), the symmetric square L-function $L(s, \pi, \mathrm{sym}^2)$ (which is of degree 6) is holomorphic; more precisely, one sees that the product $\zeta(3s - 1) L(s, \pi, \mathrm{sym}^2)$ is holomorphic except at $s = 1, 2/3$.

As the results of this section illustrate, the multiple Dirichlet series that continue to a product of complex planes are ready-made for establishing distribution results via contour integration. Though some of the results above are stated over $\mathbb{Q}$, in fact the method of multiple Dirichlet series applies over a general global field containing sufficiently many roots of unity; thus such mean value theorems may be established without being constrained by the proliferation of Gamma factors in higher degree extensions. The most natural theorems to prove involve sums of L-functions times weighting factors $a(s, \pi, d)$.

3.7. Determination of automorphic forms by twists of critical values. An additional application of multiple Dirichlet series, reflecting the power of the method, concerns the determination of an automorphic form by means of its twisted L-values.

A special case of one of the results in the paper of Luo and Ramakrishnan ([**43**]) is

Theorem [**43**] *Let f, g be two Hecke newforms for a congruence subgroup of $SL_2(\mathbb{Z})$. Suppose there exists a nonzero constant c s.t.*

$$L\left(\tfrac{1}{2}, f \otimes \chi_d\right) = cL\left(\tfrac{1}{2}, g \otimes \chi_d\right)$$

for all quadratic characters χ_d. Then $f = cg$.

This theorem has an application to a question of Kohnen: let g_1, g_2 be two newforms in the Kohnen subspace $S^+_{k+\frac{1}{2}}$ with Fourier coefficients $b_1(n), b_2(n)$ respectively. Suppose

$$b_1^2(|D|) = b_2^2(|D|)$$

for almost all fundamental discriminants with $(-1)^k D > 0$. Then $g_1 = \pm g_2$, i.e. you can't just switch some of the signs of the coefficients and get another eigenform. The proof uses Waldspurger's formula relating the square of $b_j(|D|)$ to a suitable multiple of a twisted central value. A similar theorem holds for central derivatives in the case of negative root number ([**44**]). By the theorem of Gross-Zagier, this allows one to determine an elliptic curve by heights of Heegner points.

Recently, the results of Luo and Ramakrishnan have been extended in two directions using the the methods of multiple Dirichlet series. First, Ji Li [**42**] extends [**43**] to π_1, π_2 cuspidal automorphic representations of $GL_2(\mathbb{A}_K)$, for K an arbitrary number field. Secondly, Chinta and Diaconu [**19**] extend [**43**] to symmetric squares of cusp forms on $GL_2(\mathbb{A}_\mathbb{Q})$.

Both of these theorems are proved by considering twisted averages of twists of central L-values. The result of J. Li should also extend to cover the case of determining π by twisted central derivatives. Over a number field, the averaging method employed by [**43**] (originating in the work of Iwaniec [**40**] and Murty-Murty [**47**]) runs into complications.

By contrast with J. Li's result, the result of [**19**] is valid only over $\mathbb{Q}$. This is because the authors need to use the bound

$$\sum_{|d|<x} \left| L\left(\tfrac{1}{2}, \pi \otimes \chi_d\right) \right| <<_\epsilon x^{5/4+\epsilon} \tag{3.30}$$

for π an automorphic form on GL(3). Of course,

$$\sum_{|d|<x} \left| L\left(\tfrac{1}{2}, \pi \otimes \chi_d\right) \right| <<_\epsilon x^{1+\epsilon}$$

is expected because of the Lindelöf conjecture but this is far out of reach. The proof in [**19**] of the bound (3.30) is valid only over $\mathbb{Q}$, because of an appeal to a character sum estimate of Heath-Brown [**36**]. It would be of great interest to see what types of bounds can be proved over an arbitrary number field.

4. Higher Twists

In this Section we discuss higher twists. The situation here is different from the quadratic twist case due to epsilon factors.

Let π be an automorphic representation of $GL(r)$ over given base field, and for the moment let $L(s,\pi)$ denote its standard complete L-function. Then $L(s,\pi)$ satisfies a functional equation

$$L(s,\pi) = \epsilon(s,\pi)\, L(1-s,\tilde{\pi}).$$

To study a sum of twists via Hartogs' principle/Bochner's theorem, the relationship between $\epsilon(s,\pi)$ and $\epsilon(s,\pi\times\chi_d^{(n)})$ is needed. The quotient is a power of the conductor, which is essentially the square-free part of n, times the quotient at $s = 1/2$. This last factor is essentially the r-th power of a Gauss sum of order n, $G(\chi)^r$, that has been normalized to have absolute value 1. (More precisely, this is true after sieving, so it is more convenient to work with a finite dimensional vector space of multiple Dirichlet series; see Fisher-Friedberg [**29**] for a discussion of this point in a classical language and Brubaker-Bump [**5**] for a discussion which is adelic in nature.)

Because of this crucial change, the heuristic that describes the quadratic twist case is not useful. In fact, after a functional equation one obtains a *new* mutliple Dirichlet series—not $Z(s, w)$, but a series whose weight factors involve n-th order Gauss sums. A similar situation occurs if one interchanges and then applies a functional equation. Moreover, these two operations need not commute (even ignoring scattering matrix and bad prime considerations)! To use the convexity methods of Section 3.2, one is then led to consider several different families of multiple Dirichlet series that are linked by functional equations. We discuss two cases in detail (n-fold twists of GL(1) and cubic twists of GL(2). This is followed by a discussion of the nonvanishing of n-th order twists of a GL(2) automorphic L-function for arbitrary n. Though the sum of twisted L-functions $Z(s, w)$ has not been continued to $\mathbb{C}^2$, a variation on the method of double Dirichlet series gives an interesting result.

4.1. n-Fold Twists of GL(1). The study of the sum of the n-fold twists of a given Hecke character was carried out by Friedberg, Hoffstein and Lieman [**32**]. One obtains two different families of multiple Dirichlet series: the n-th order twists of the original L-function $\sum L(s, \xi\chi_d^{(n)})a(s, \xi, d)|d|^{-w}$ and a multiple Dirichlet series built up from infinite sums of n-th order Gauss sums. The second series is obtained from the first by use of the functional equation for $L(s, \xi\chi_d^{(n)}$ followed by an interchange of summation. But these latter sums arise as the Fourier coefficients of Eisenstein series on the n-fold cover of GL(2), and they can thus be controlled by using the theory of metaplectic Eisenstein series. In particular, they satisfy a functional equation of their own, even though they are not Eulerian! To keep this paper to manageable length, we do not give many details; we will supply them in the more complicated case of GL(2) below. We remark that automorphic methods, which could be for the most part avoided in the quadratic twist case, seem unavoidable in many problems involving n-th order twists for $n > 2$.

In the case at hand, the continuation of the two families of double Dirichlet series to $\mathbb{C}^2$ is established from Bochner's theorem. Note that earlier we mentioned that such a sum could be approached by an integral of an Eisenstein series on the n-fold cover of GL(n). Thus the Hartogs/Bochner-based method allows one to replace the use of Eisenstein series on the n-fold cover of GL(n) with the use of Eisenstein series on the n-fold cover of GL(2), which are considerably simpler. We shall see a similar reduction to GL(2) in the work on Weyl group multiple Dirichlet series that is discussed in [**7**].

Let us also note that Brubaker and Bump ([**6**], in this volume) have obtained the double Dirichlet series discussed in this section as residues of Weyl group multiple Dirichlet series, and have shown that their functional equations may be understood as a consequence of this fact. They take $n = 3$ for convenience, but (as they explain) one should have such a realization for all $n \geq 3$.

4.2. Cubic Twists of GL(2).

4.2.1. *The main result.* The double Dirichlet series coming from cubic twists of an automorphic representation on GL(2) was continued by Brubaker, Friedberg and Hoffstein [**10**]. Let $K = \mathbb{Q}(\sqrt{-3})$. For $d \in O_K$, $d \equiv 1 \bmod 3$ let $|d|$ denote the absolute norm of d. Let $P(s; d)$ denote a certain Dirichlet polynomial defined in [**10**]; $P(s;d)$ depends on π but we suppress this from the notation. $P(s;d)$ is a complicated object, but has the properties that if one factors $d = d_1 d_2^2 d_3^3$ with each $d_i \equiv 1 \bmod 3$, d_1 square-free, $d_1 d_2^2$ cube-free, then $P(s;d) = 1$ if $d_3 = 1$. Also for fixed d_1, d_2, the sum

$$\sum_{d_3 \equiv 1 \mod 3} \frac{P(s; d_1 d_2^2 d_3^3)}{|d_3|^{3w}}$$

converges absolutely for $\Re w > 1/2$ and $\Re s \geq 1/2$.

The main theorem of [**10**] is:

THEOREM 4.1. *Let $\pi = \otimes \pi_v$ be an automorphic representation of $GL(2, \mathbb{A}_K)$ such that $L(s, \pi, \chi)$ is entire for all Hecke characters χ such that $\chi^3 = 1$. Let S be a finite set of primes including the archimedean prime and the primes dividing $2, 3$ and the level of π. Then, for any sufficiently large positive integer k, the asymptotic formula*

$$\sum_{|d|<X} L_S(s, \pi, \chi^{(3)}_{d_1 d_2^2}) P(s;d) \left(1 - \frac{|d|}{X}\right)^k \sim \frac{1}{k+1} c^{(3)}(s, \pi) X$$

holds for any s with $\Re s \geq 1/2$. The constant $c^{(3)}(s,\pi)$ is non-zero, and is given by

$$c^{(3)}(s,\pi) = c_S L_S(3s, \pi, \vee^3) \zeta_S(6s) \zeta_S(6s+1)^{-1} \prod_{p \notin S} (1 - \gamma_p^3 |p|^{-3s-1})(1 - \delta_p^3 |p|^{-3s-1}),$$

where ζ_S denotes the Dedekind zeta function of K with the Euler factors at the places in S removed, γ_p, δ_p are the Satake parameters of the representation π_p, and c_S is a non-zero constant.

An immediate consequence of this, the convergence of the basic sum, and the usual convexity bound for $L(1/2, \pi, \chi^{(3)}_{d_1 d_2^2})$ is

COROLLARY 4.2. *Let π be as in (4.1) Then there exist infinitely many cube-free d such that $L(1/2, \pi, \chi^{(3)}_d) \neq 0$. More precisely, let $N(X)$ denote the number of such d with $|d| \leq X$. Then for any $\epsilon > 0$, $N(X) \gg X^{1/2-\epsilon}$.*

We sketch the proof, which is somewhat involved. Define the multiple Dirichlet series

$$Z_1(s,w) = \sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{L_S(s, \pi, \chi^{(3)}_{d_1 d_2^2}) P(s;d)}{|d|^w}.$$

(Here the sum is over all $d \in O_K$ with $d \equiv 1 \mod 3$ and $\mathrm{ord}_v(d) = 0$ for all finite $v \in S$.) This series converges absolutely for $\Re(s), \Re(w) > 1$. Our goal is to establish the continuation of this function to a larger region. Let

$$Z^*(s,w) = Z_1(s,w)\, \zeta_S(6s+6w-5)\, \zeta_S(12s+6w-8) \times$$
$$\prod_{p \notin S} (1 - \gamma_p^3 |p|^{2-3s-3w})^{-1} (1 - \delta_p^3 |p|^{2-3s-3w})^{-1},$$

where γ_p, δ_p are the Satake parameters of the representation π_p. In fact, Brubaker, Friedberg and Hoffstein show that $Z^*(s,w)$ has a meromorphic continuation to the half plane $\Re(s+w) > 1/2$ and is analytic in this region except for polar lines at $w = 1$, $w = 0$, $w = 5/3 - 2s$, $w = 3/2 - 2s$, $w = 4/3 - 2s$, $w = 7/6 - s$, $w = 1 - s$,

$w = 5/6 - s$. (With a little more work, they could establish continuation to $\mathbb{C}^2$; see below.) They also show that the residue at $w = 1$ satisfies

$$\mathrm{Res}_{w=1} Z^*(s, w) = c_S\, L_S(3s, \pi, sym^3)\, \zeta_S(6s)\, \zeta_S(12s - 2)$$

and is an analytic function of s for $\Re s > -1/2$, except possibly at the points $s = 1/3, 1/4, 1/6, 0$, which require a more detailed analysis. The properties of the symmetric cube L-series have been completely described by Kim and Shahidi.

4.2.2. *The first two series and the first functional equation.* This step is based on the exact functional equation for the cubically-twisted L-series. Write $d = d_1 d_2^2 d_3^3$ as above. Ignoring bad primes such as those dividing the level of π and the infinite place, $L(s, \pi, \chi^{(3)}_{d_1 d_2^2})$ has a functional equation of the form

$$L(s, \pi, \chi^{(3)}_{d_1 d_2^2}) \to \epsilon_\pi G(\chi^{(3)}_{d_1 d_2^2})^2 L(1 - s, \tilde{\pi}, \bar{\chi}^{(3)}_{d_1 d_2^2}) |d_1 d_2|^{1-2s}.$$

Here $\tilde{\pi}$ denotes the contragredient of π, ϵ_π (the central value of the usual epsilon-factor for π) has absolute value 1 and $G(\chi^{(3)}_d)$ is the usual Gauss sum associated to $\chi^{(3)}_d$, normalized to have absolute value 1. The crucial factor $|d_1 d_2|^{1-2s}$ arises as part of the epsilon-factor of the twisted L-function since $\pi \otimes \chi^{(3)}_d$ is ramified at the primes dividing $d_1 d_2$. This functional equation gives rise to a functional equation for the double Dirichlet series Z_1, reflecting $Z_1(s, w)$ into a second double Dirichlet series

$$Z_6(s, w) = \sum \frac{L_S(s, \tilde{\pi}, \bar{\chi}^{(3)}_{d_1 d_2^2})\, G(\chi^{(3)}_{d_1 d_2^2})^2\, P(1 - s; d_1 d_2^2 d_3^3)\, |d_2 d_3^3|^{1-2s}}{|d_1 d_2^2 d_3^3|^w}.$$

More precisely, the functional equation above induces a transformation relating $Z_1(s, w)$ to $Z_6(1-s, w+2s-1)$. (The exact transformation is somewhat complicated due to bad primes.)

4.2.3. *The second functional equation.* Next we study the series $Z_6(s, w)$ itself. The appearance of $G(\chi^{(3)}_{d_1 d_2^2})^2$, the square of a cubic Gauss sum, introduces, *via the Hasse-Davenport relation*, a conjugate 6-th order Gauss sum. However, the Fourier coefficients of Eisenstein series on the 6-fold cover of GL(2) may be written as sums of Gauss sums

$$\sum_{d \equiv 1 \mod 3, (d,S)=1} \frac{G^{(6)}(m, d)}{|d|^w},$$

and accordingly series of this type possess a functional equation in w. One may show, using this functional equation, that $Z_6(s, w)$ possesses a functional equation as $(s, w) \to (s + 2w - 1, 1 - w)$, transforming into itself.

4.2.4. *The third series and the third functional equation.* The authors of [**8**] next show that the order of summation in $Z_1(s, w)$ written as a doubly-indexed Dirichlet series can be interchanged, leading to an expression of the form

$$Z_1(s, w) = \sum \frac{L_S(w, \chi^{(3)}_{m_1 m_2^2})\, Q(w; m_1 m_2^2 m_3^3)}{|m_1 m_2^2 m_3^3|^s},$$

where Q is once again a specific Dirichlet polynomial depending on π and the L-series on the right are Hecke L-series. Applying the functional equation in w for the Hecke L-series they are led to introduce the third double Dirichlet series

$$Z_3(s,w)=\sum\frac{L_S(w,\bar\chi^{(3)}_{m_1m_2^2})\,G(\chi^{(3)}_{m_1m_2^2})\,Q(1-w;m_1m_2^2m_3^3)\,|m_2m_3^3|^{1/2-w}}{|m_1m_2^2m_3^3|^s}.$$

The functional equation for the Hecke L-series induces a transformation relating $Z_1(s,w)$ to $Z_3(s+w-1/2,1-w)$.

Once again, the series Z_3 may be studied using metaplectic Eisenstein series. Indeed, after an interchange of the order of summation, this series is a sum of cubic twists of Rankin-Selberg convolutions of π with the theta function on the 3-fold cover of GL(2). (Recall that this function is the residue of an Eisenstein series on the 3-fold cover of GL(2); see Patterson's Crelle paper.) From the meromorphic continuation of the twisted Rankin-Selberg convolutions one may then deduce a corresponding continuation for Z_3.

4.2.5. *Applying Bochner's Theorem.* One may now apply Bochner's theorem to obtain the continuation of these 3 functions. The functions $Z_1(s,w)$ and $Z_6(s,w)$ have overlapping regions of absolute convergence. If the functional equation interchanging $Z_1(s,w)$ and $Z_6(s,w)$ is used several times, the convexity principle for several complex variables applied to the union of translates of these regions implies an analytic continuation of $Z_1(s,w)$ and $Z_6(s,w)$ to the half plane $\Re(w+s)>3/2$. The relations with $Z_3(s,w)$ then imply an analytic continuation to the half plane $\Re(w+s)>1/2$, which is what is required for the applications.

Remarks:

(1) A further functional equation, transforming $Z_3(s,w)$ into itself as $(s,w)\to(1-s,w+4s-2)$, can be proved. This then allows an analytic continuation of all three double Dirichlet series to $\mathbb{C}^2$. This also gives rise to a group of functional equations which is non-abelian and of order 384. These computations have not been written down in detail.

(2) As mentioned above, in the quadratic twist case the double Dirichlet series for $r=1,2,3$ can be identified, up to a finite number of places, with certain integral transforms of metaplectic Eisenstein series. In the case at hand, although there is no known way to construct the double Dirichlet series as a similar integral transform (or as a Rankin-Selberg convolution), there is a natural candidate attached to the cubic cover of G_2, and it is possible that the complicated formulas of [**10**] reflect in a certain sense combinatorial issues arising from that group.

(3) One may also obtain a mean value result for the product of two Hecke L-functions in different variables when they are simultaneously twisted by cubic characters. This was accomplished by Brubaker [**3**] in his Brown University doctoral dissertation.

4.3. The nonvanishing of n-th order twists of a GL(2) form. Let E be an elliptic curve defined over a number field K. The behavior of the rank of the L-rational points $E(L)$ as L varies over some family of algebraic extensions of K is a problem of fundamental interest. The conjecture of Birch and Swinnerton-Dyer provides a means to investigate this problem via the theory of automorphic L-functions.

Assume that the L-function of E coincides with the L-function $L(s,\pi)$ of a cuspidal automorphic representation of GL(2) of the adele ring $\mathbb{A}_K$. Let L/K be a finite cyclic extension and χ a Galois character of this extension. Then the conjecture of Birch and Swinnerton-Dyer equates the rank of the χ-isotypic component $E(L)^\chi$ of $E(L)$ with the order of vanishing of the twisted L-function $L(s, \pi \otimes \chi)$ at the central point $s = \frac{1}{2}$. In particular, the χ-component $E(L)^\chi$ is finite (according to the conjecture) if and only if the central value $L(\frac{1}{2}, \pi \otimes \chi)$ is non-zero.

Thus it becomes of arithmetic interest to establish non-vanishing results for central values of twists of automorphic L-functions by characters of finite order. For quadratic twists this problem has received much attention in recent years. Using the method of multiple Dirichlet series, the paper [**4**] addresses this question for twists of higher order.

THEOREM 4.3. [**4**] *Fix a prime integer $n > 2$, a number field K containing the n^{th} roots of unity, and a sufficiently large finite set of primes S of K. Let π be a self-contragredient cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$ which has trivial central character and is unramified outside S. Suppose there exists an idèle class character χ_0 of K of order n unramified outside S such that*

$$L(\tfrac{1}{2}, \pi \otimes \chi_0) \neq 0.$$

Then there exist infinitely many idèle class characters χ of K of order n unramified outside S such that

$$L(\tfrac{1}{2}, \pi \otimes \chi) \neq 0.$$

Fearnley and Kisilevsky have proven a related result for the L-function $L(s, E)$ of an elliptic curve defined over $\mathbb{Q}$. In [**28**] they show that if the algebraic part $L^{\mathrm{alg}}(\frac{1}{2}, E)$ of the central L-value is nonzero mod n, then there exist infinitely many Dirichlet characters χ of order n such that $L(\frac{1}{2}, E, \chi) \neq 0$. If $L(\frac{1}{2}, E) \neq 0$ then the hypothesis $L^{\mathrm{alg}}(\frac{1}{2}, E) \not\equiv 0$ mod n is satisfied for all sufficiently large primes n. Note the necessity of the assumption $L\left(\frac{1}{2}, \pi\right) \neq 0$ in both [**28**],[**4**]. The theorem should be true without this assumption. In fact, "almost all" twists should be nonzero when $n > 2$. (See e.g. [**23**] where a random matrix model is given for predicting the frequency of vanishing twists.)

Another related result is the beautiful theorem of Diaconu and Tian, [**26**].

THEOREM 4.4. [**26**]*Let p be a prime number, F a totally real field of odd degree s.t. $[F(\mu_p) : F] = 2$. Let W_δ be the twisted Fermat curve*

$$W_\delta : x^p + y^p = \delta.$$

Then there exist infinitely many $\delta \in F^\times / F^{\times p}$ for which W_δ has no F-rational solutions.

The proof of this result is based on Zhang's extension of the Gross-Zagier formula to totally real fields and on Kolyvagin's technique of Euler systems. Then, a double Dirichlet series is used to show that a certain family of twisted L-series has nonvanishing central value infinitely often.

5. A Rational Function Field Example

The goal of this section is to work out in detail the example of a double Dirichlet series over a function field. Many key features of the theory of multiple Dirichlet series appear already in this example and several technical complications that occur in the more general cases are not present here. Among the advantages of working over the rational functional field are that the rational function field has class number one, quadratic reciprocity is particularly simple in this setting and there is only one bad place.

5.1. The rational function field. We begin by setting up some notation and reviewing some basic facts about the zeta function of the rational function field, quadratic reciprocity and Dirichlet L-functions. For proofs of these facts see, for example, Moreno [**46**] or Rosen [**50**].

Let q be an odd prime power, congruent to 1 mod 4. (This congruence condition will simplify the statement of quadratic reciprocity.) Let $\mathbb{F}_q[t]$ be the polynomial ring in t with coefficients in the finite field $\mathbb{F}_q$. This is a principal ideal domain. The nonzero prime ideals of $\mathbb{F}_q[t]$ are generated by irreducible polynomials. We let $\mathbb{F}_q(t)$ denote the quotient field. Define the norm function $\mathbf{N}(f) = |f| = q^{\deg f}$ for $f \in \mathbb{F}_q[t]$.

The zeta function of the ring $\mathbb{F}_q[t]$ is defined either by an Euler product or a Dirichlet series

$$\zeta(s) = \prod_{\substack{P\in\mathbb{F}_q[t] \\ P \ irred,monic}} \left(1 - \frac{1}{|P|^s}\right)^{-1} = \sum_{\substack{f\in\mathbb{F}_q[t] \\ f \ monic}} \frac{1}{|f|^s}.$$

The equality of the product and sum above is a reformulation of the fact that $\mathbb{F}_q[t]$ is a unique factorization domain. As there are q^n monic polynomials of degree n, we may sum a geometric series to get a very explicit expression for the zeta function:

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{\#\text{ of monic polys of deg } n}{q^{ns}} = \frac{1}{1-q^{1-s}}.$$

This zeta function satisfies a functional equation. Define the completed zeta function to be

$$\zeta^*(s) = \frac{1}{1-q^{-s}}\zeta(s).$$

Then

$$\zeta^*(s) = q^{2s-1}\zeta^*(1-s).$$

Remark The term $(1-q^{-s})^{-1}$ in the completed zeta function corresponds to the contribution from the place at infinity. In what follows, we will find it convenient to deal with this place separately.

We now turn to defining the quadratic residue symbol and quadratic L-functions. For f an irreducible, monic polynomial in $\mathbb{F}_q[t]$, define

$$\chi_f(g) = \left(\frac{f}{g}\right) = g^{(|f|-1)/2}(\bmod f).$$

Thus $\chi_f(g) = \pm 1$ for f, g relatively prime. If f_1, f_2 are two monic polynomials such that f_1f_2 is square-free, we define $\chi_{f_1f_2} = \chi_{f_1}\chi_{f_2}$. In this way χ_f now makes sense whenever f is monic and square-free. The quadratic residue symbol has the following fundamental reciprocity property:

Quadratic Reciprocity Let $f, g \in \mathbb{F}_q[t]$ be monic, square-free and relatively prime. Then

$$\left(\frac{f}{g}\right) = (-1)^{\frac{q-1}{2}\deg f \deg g}\left(\frac{g}{f}\right).$$

Note that in the case where q is congruent to 1 mod 4 (as we will henceforth assume) the sign on the right is always $+1$.

For f monic and square-free, define the L-series associated to the quadratic residue symbol χ_f by

$$\begin{aligned} L(s,\chi_f) &= \prod_{P \nmid f} \left(1 - \frac{\chi_f(P)}{|P|^s}\right)^{-1} \\ &= \sum_{\substack{g \ monic \\ (g,f)=1}} \frac{\chi_f(g)}{|g|^s} \end{aligned}$$

and the completed L-series by

$$L^*(s,\chi_f) = \begin{cases} \frac{1}{1-q^{-s}} L(s,\chi_f) & \text{if deg } f \text{ even} \\ L(s,\chi_f) & \text{if deg } f \text{ odd.} \end{cases}$$

The completed L-function satisfies the functional equation

$$L^*(s,\chi_f) = \begin{cases} q^{2s-1}|f|^{1/2-s} L^*(1-s,\chi_f) & \text{if deg } f \text{ even} \\ q^{2s-1}(q|f|)^{1/2-s} L^*(1-s,\chi_f) & \text{if deg } f \text{ odd.} \end{cases}$$

Remarks

(1) The term raised to the power $\frac{1}{2} - s$ is the conductor of the character χ_f. If the degree of f is odd, the conductor of χ_f is $q|f|$ because of an additional ramification at the place at infinity.
(2) As in the case of the zeta function, the functional equations look simpler when the Euler facter at infinity is included. However, for our purposes, we will find it convenient to leave it out. Similarly, over a number field, a finite number of places need to be dealt with separately.

5.2. The GL(1) quadratic double Dirichlet Series. In this section we will construct the multiple Dirichlet series in two variables associated to the sum of quadratic (GL(1)) L-functions. We will continue to work over the rational function field, however all of the local computations we do in constructing the weighting polynomials will be valid for any global field.

The double Dirichlet series we wish to construct is roughly of the form

$$Z(s,w) \approx \sum_{\substack{f \in \mathbb{F}_q[t] \\ f \ monic}} \frac{L(s,\chi_f)}{|f|^w} = \sum \sum \frac{\left(\frac{f}{g}\right)}{|f|^w |g|^s}.$$

For maximal symmetery, we wish to sum over all f and g monic and nonzero, however our quadratic residue symbol $\chi_f(g)$ only makes sense when fg is square-free. We want to define the quadratic residue symbols in such a way that

- the definition agrees with our old definition when fg is square-free
- summing over g (resp. f) produces an L-series in s (resp. w) with an Euler product and satisfying the "right" functional equation

It turns out that there is a unique way to do this. We will explain in the following section what "right" means. Basically, functional equations of the individual $L(s,\chi_f)$'s should induce a functional equation in $Z(s,w)$.

The precise definition of the double Dirichlet series will be

$$Z(s,w) = \sum_f \sum_g \frac{\chi_{f_0}(\hat{g}) b(g,f)}{|f|^w |g|^s}$$

where

- f_0 is the square-free part of f,
- $\hat{g}$ is the part of g relatively prime to f, and
- the coefficients $b(g,f)$ should be multiplicative and chosen to ensure the proper functional equations.

5.2.1. *Weighting polynomials and the coefficients* $b(g,f)$. We now turn to the definition of the weighting coefficient $b(g,f)$. These coefficients will be multiplicative in the sense that

$$b(g,f) = \prod_{\substack{P^\alpha || g \\ P^\beta || f}} b(P^\alpha, P^\beta).$$

We also require that $b(1,f) = b(f,1) = 1$ for all f.

Therefore

$$L(s, \hat{\chi}_f) := \sum_g \frac{\chi_{f_0}(g) b(g,f)}{|g|^s}$$

has the Euler product

$$\prod_P \left(\sum_{k=0}^{\infty} \frac{\chi_{f_0}(\hat{P}^k) b(P^k, f)}{|P|^{ks}} \right) = L(s, \chi_{f_0}) \mathcal{Q}_f(s),$$

say, where $\mathcal{Q}_f(s)$ is a finite Euler product supported in the primes dividing f to order greater than 1. We can describe $\mathcal{Q}_f$ explicitly in terms of the weighting coefficients. Let $f = f_0 f_1^2 f_2^2$, where f_0 is squarefree and f_2 is relatively prime to $f_0 f_1$. Then

$$\mathcal{Q}_f(s) = \prod_{P^\alpha || f_1} \mathcal{Q}_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta || f_2} \mathcal{Q}_{P^{2\beta}}(s, \chi_{f_0}(P)) \tag{5.1}$$

where

$$\mathcal{Q}_{P^{2\alpha+1}}(s) = \sum_{k=0}^{\infty} \frac{b(P^k, P^{2\alpha+1})}{|P|^{ks}}, \text{ and} \tag{5.2}$$

$$\mathcal{Q}_{P^{2\beta}}(s, \chi_{f_0}(P)) = (1 - |P|^{-s}) \sum_{k=0}^{\infty} \frac{\chi_{f_0}(P^k) b(P^k, P^{2\beta})}{|P|^{ks}}.$$

We want $L(s, \hat{\chi}_f)$ to satisfy the same form of functional equation as $L(s, \chi_{f_0})$. Namely, we want

$$L(s, \hat{\chi}_f) = \begin{cases} q^{2s-1} \frac{1-q^{-s}}{1-q^{s-1}} |f|^{1/2-s} L(1-s, \hat{\chi}_f) & \text{if } \deg f \text{ even} \\ q^{2s-1} (q|f|)^{1/2-s} L(1-s, \hat{\chi}_f) & \text{if } \deg f \text{ odd.} \end{cases} \tag{5.3}$$

It follows that the weighting polynomials must satisfy the functional equation

$$\mathcal{Q}_f(s) = \left| \frac{f}{f_0} \right|^{\frac{1}{2}-s} \mathcal{Q}_f(1-s).$$

This is motivated by the desire to have an $(s,w) \mapsto (1-s, s+w-\frac{1}{2})$ functional equation in the double Dirichlet series $Z(s,w)$. There is an identical requirement for the sums

$$L(w, \hat{\chi}_g) := \sum_f \frac{\chi_{g_0}(f) b(g,f)}{|f|^w},$$

translating into an $(s,w) \mapsto (s+w-\frac{1}{2}, 1-w)$ functional equation for the double Dirichlet series. For simplicity, we stipulate that $b(f,g) = b(g,f)$.

As we will describe below, it turns out that these conditions, i.e. multiplicativity and functional equations for weighting polynomials, determine the the coefficients $b(g,f)$ uniquely.

Examples Let P be an irreducible polynomial of norm p

(i) $\mathcal{Q}_1(s) = \mathcal{Q}_P(s) = 1$
(ii) $\mathcal{Q}_{P^2}(s) = 1 - \frac{1}{p^s} + \frac{p}{p^{2s}}$
(iii) $\mathcal{Q}_{P^3}(s) = 1 + \frac{p}{p^{2s}}$
(iv) $\mathcal{Q}_{P^4}(s) = 1 - \frac{1}{p^s} + \frac{p}{p^{2s}} - \frac{p}{p^{3s}} + \frac{p^2}{p^{4s}}$

5.2.2. *A generating function.* Let us reformulate the functional equations of the weighting polynomials in terms of the coefficients $b(P^k, P^l)$.

Fix an irreducible polynomial P of norm p and let $x = p^{-s}, y = p^{-w}$. Construct the generating series

$$H(x,y) = \sum_{k,l=0}^{\infty} b(P^k, P^l) x^k y^l.$$

Summing over one index (say k) while leaving the other fixed, we get the P-part of $L(s, \hat{\chi}_{P^l})$:

$$\sum_k b(P^k, P^l) x^k = \begin{cases} \mathcal{Q}_{P^l}(x) & \text{if } l \text{ odd} \\ \frac{1}{1-x} \mathcal{Q}_{P^l}(x) & \text{if } l \text{ even.} \end{cases} \tag{5.4}$$

Recall that the weighting polynomials satisfy

$$\mathcal{Q}_{P^{2l+i}}(x) = (x\sqrt{p})^{2l} \mathcal{Q}_{P^{2l+i}}\left(\frac{1}{px}\right)$$

for $i = 0, 1$.

By virtue of the functional equations satisfied by the $\mathcal{Q}$, the generating series $H(x,y)$ will satisfy a certain functional equation. We describe this now, together with the limiting behavior and x, y symmetry of H.

(A1) $H(x,y) = H(y,x)$
(A2) $H(x,0) = 1/(1-x)$
(A3) The auxiliary functions

$$\begin{aligned} H_0(x,y) &:= (1-x)\left[H(x,y) + H(x,-y)\right], \\ H_1(x,y) &:= \frac{1}{y}\left[H(x,y) - H(x,-y)\right] \end{aligned}$$

are invariant under the transformation

$$(x,y) \mapsto \left(\tfrac{1}{px}, xy\sqrt{p}\right).$$

The H_0 and H_1 isolate the the cases l even and l odd. This is necessary because, as exhibited in (5.4), the weighting polynomials for l even and l odd have slightly different expressions in terms of the generating series. The functional equations above can be more cleanly written in vector notation as

$$\mathbf{H}(x,y) := \begin{pmatrix} H(x,y) \\ H(x,-y) \\ H(-x,y) \\ H(-x,-y) \end{pmatrix} = \Phi(x,y) \mathbf{H}\left(\tfrac{1}{px}, xy\sqrt{p}\right)$$

where Φ is a 4×4 scattering matrix.

Another way to think of this is that in order to get precise functional equations for the double Dirichlet series $Z(s, w)$, it is necessary to consider also twists of the form

$$Z(s, w; \psi) = \sum \frac{L(s, \hat{\chi}_f)\psi(f)}{|f|^w}$$

by the idele class character $\psi(f) = (-1)^{\deg f}$. Then, taking linear combinations with the untwisted series, we can isolate the sum to be over f in congruence classes in which the Γ-factor of the functional equation (5.3) is constant. Over a number field (or function field of higher genus), in order to effect the interchange of summation, one needs to do something similar to isolate congruence classes on which the Hilbert symbol is constant.

5.2.3. *The generating function $H(x, y)$ and functional equations of $Z(s, w)$.* There is a unique power series in x, y satisfying **A1, A2** and **A3**:

$$H(x, y) = \frac{1 - xy}{(1 - x)(1 - y)(1 - px^2y^2)}. \tag{5.5}$$

With the $b(P^k, P^l)$ defined implicitly by the above generating series, the double Dirichlet series $Z(s, w)$ will satisfy functional equations

$$(s, w) \mapsto (1 - s, w + s - \tfrac{1}{2})$$
$$(s, w) \mapsto (s + w - \tfrac{1}{2}, 1 - w).$$

(To be more precise, the vector consisting of $Z(s, w)$ and twists by the idèle class character ψ defined above will satisfy vector-valued functional equations with a scattering matrix.) These two functional equations generate a group G, isomorphic to the dihedral group of order 6. The double Dirichlet series $Z(s, w)$ may then be analytically continued by the convexity arguments of Section 3.

We conclude this subsection by showing how the expression (3.15) for the $GL(1)$ correction polynomials can be recovered from the generating function $H(x, y)$. For simplicity, we take π in (3.15) to be trivial. Then, in our notation, $\mathcal{Q}_f(s) = a(s, \pi, f)$. Combining the expression (5.2) for the P-part of $\mathcal{Q}_f$ with (5.5) we can now compute

$$\sum_{k=0}^{\infty} \frac{\mathcal{Q}_{P^{2k+1}}(s)}{|P|^{2kw}} = \frac{1}{(1 - |P|^{-2w})(1 - |P|^{1-2s-2w})}$$

and

$$\sum_{l=0}^{\infty} \frac{\mathcal{Q}_{f_0 P^{2l}}(s)}{|P|^{2lw}} = \frac{1 - \chi_{f_0}(P)|P|^{-s-2w}}{(1 - |P|^{-2w})(1 - |P|^{1-2s-2w})}$$

for f_0 squarefree and relatively prime to P. Therefore

$$\begin{aligned}
&\sum_{\substack{e \in \mathbb{F}_q[t] \\ e \text{ monic}}} \frac{\mathcal{Q}_{f_0 e^2}(s)}{|e|^{2w}} \\
&= \left(\prod_{P | f_0} \sum_{k=0}^{\infty} \frac{\mathcal{Q}_{P^{2k+1}}(s)}{|P|^{2kw}} \right) \cdot \left(\prod_{(P, f_0)=1} \sum_{l=0}^{\infty} \frac{\mathcal{Q}_{P^{2l}}(s, \chi_{f_0}(P))}{|P|^{2lw}} \right) \\
&= \frac{\zeta(2w)\zeta(2s + 2w - 1)}{L(s + 2w, \chi_{f_0})}.
\end{aligned}$$

Expressing the final quotient of L-functions as a Dirichlet series in w and extracting the coefficient of $|e|^{2w}$ gives

$$\mathcal{Q}_{f_0e^2}(s) = \sum_{e_1e_2e_3=e} \mu(e_1)\chi_{f_0}(e_1)|e_1|^{-s}|e_3|^{1-2s}.$$

This is precisely (3.16).

5.3. Application: mean values of L-functions. Analytic properties of a Dirichlet series can often be translated (via contour integration or Tauberian theorems) into information about partial sums of the coefficients of the series.

For example, let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a holomorphic function of s for $\Re(s) > \sigma \in \mathbb{R}$. Suppose that $F(s)$ has a pole of order $r+1$ at $s = \sigma$ with leading term c and is otherwise holomorphic for $\Re(s) > \sigma - \epsilon$. Then, under some mild growth restrictions on F,

$$\sum_{n<X} a_n \sim \frac{c}{r!} X^{\sigma} (\log X)^r$$

as $X \to \infty$.

One application of the theory of multiple Dirichlet series is to deduce mean value properties for special values of L-functions from the analytic properties of a multiple Dirichlet series.

To describe this in this simple example, we first need to compute the poles and residues of $Z(s, w)$.

5.3.1. *Poles of* $Z(s, w)$. The double Dirichlet series

$$Z(s, w) = \sum_{f} \frac{L(s, \hat{\chi}_f)}{|f|^w}$$

has an obvious pole at $s = 1$ coming from the pole of the ζ-function when f is a perfect square. Translating by the group G of functional equations gives the complete set of polar divisors of $Z(s, w)$:

$$s = 1, w = 1, s + w = 3/2.$$

(The other translates of $s = 1$ by the group G correspond to poles of the gamma function.)

5.3.2. *The residue at* $w = 1$. We will use the expression

$$Z(s, w) = \sum_{g} \frac{L(w, \chi_{g_0})\mathcal{Q}_g(w)}{|g|^s}$$

and knowledge of the weighting polynomials to compute the residue of $Z(s, w)$ at $w = 1$.

The numerator $L(w, \chi_{g_0})\mathcal{Q}_g(w)$ of the summand has a simple pole at $w = 1$ iff g is a perfect square. In this case, the residue of $L(w, \chi_{g_0})\mathcal{Q}_g(w)$ is simply $c \cdot \mathcal{Q}_g(1)$, where c is the residue of the zeta function. Now, $\mathcal{Q}_g(w) = \prod_{P^{2\alpha}||g} \mathcal{Q}_{P^{2\alpha}}(w)$.

From the explicit computation of $H(x, y)$ we find that

$$\sum_{k=0}^{\infty} \frac{\mathcal{Q}_{P^{2k}(1)}}{p^{2ks}} = \frac{1}{1 - p^{-2s}},$$

and hence $\mathcal{Q}_{P^{2k}}(1) = 1$ for all k, P, which implies

$$\operatorname*{Res}_{w=1} Z(s,w) = R_1(s) = c\zeta(2s).$$

5.3.3. *The pole of* $Z\left(\frac{1}{2}, w\right)$ *at* $w = 1$. To compute mean values of $L\left(\frac{1}{2}, \hat{\chi}_f\right)$ we need to understand the polar structure of $Z\left(\frac{1}{2}, w\right)$ as a function of w. The location of the first pole ($w = 1$) is immediate from what we have already done. The computation of the order is a little more involved.

In a neighborhood of $\left(\frac{1}{2}, 1\right)$ the double Dirichlet series looks like

$$Z(s,w) = \frac{R_1(s)}{w-1} + \frac{R_2(s)}{w+s-\frac{3}{2}} + Y(s,w),$$

where $Y(s,w)$ is holomorphic in a neighborhood of $\left(\frac{1}{2}, 1\right)$.

Using the facts that $R_1(s)$ has a simple pole at $s = \frac{1}{2}$ and that $Z\left(\frac{1}{2}, w\right)$ is holomorphic for $w > 1$ we deduce that $R_2(s)$ must also have a simple pole at $s = \frac{1}{2}$ which cancels the pole from R_1. Therefore, we have

$$\begin{aligned} Z(s,w) = &\frac{A_1}{(w-1)(s-\frac{1}{2})} + \frac{A_2}{w-1} \\ &\qquad - \frac{A_1}{(w+s-\frac{3}{2})(s-\frac{1}{2})} + \frac{B_2}{w+s-\frac{3}{2}} + Y(s,w) \end{aligned}$$

for some constants A_1, A_2, B_2. Setting $s = \frac{1}{2}$ we conclude that

$$Z\left(\tfrac{1}{2}, w\right) = \frac{A_1}{(w-1)^2} + \frac{A_1'}{w-1} + O(1)$$

in a neighborhood of $w = 1$, where $A_1^{'} = A_2 + B_2$.

5.3.4. *Mean values of* $L\left(\frac{1}{2}, \hat{\chi}_f\right)$. By contour integration, it follows that

$$\sum_{|f|<x} L\left(\tfrac{1}{2}, \hat{\chi}_f\right) = A_1 x \log x + A_1' x + o(x)$$

as $x \to \infty$.

Since A_1 is nonzero, it follows that $L\left(\frac{1}{2}, \chi_f\right)$ is nonzero infinitely often.

5.4. Computing $Z(s,w)$**.** As noted earlier, for a function field, the group of functional equations satisfied by the double Dirichlet series $Z(s,w)$ will force it to be a rational function. In this section, our goal is the following

Goal: With $b(g,f)$ defined as above, express

$$Z(s,w) = \sum_f \sum_g \frac{\chi_{f_0}(\hat{g}) b(g,f)}{|f|^w |g|^s}$$

as a rational function of $x = q^{-s}, y = q^{-w}$.

Recall: if $f = f_0 f_1^2$ with f_0 square-free,

$$L(s, \hat{\chi}_f) = \sum_{\substack{g \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{b(g,f)\chi_{f_0}(\hat{g})}{|g|^s} = L(s, \chi_{f_0}) \mathcal{Q}_f(s).$$

Because of the functional equation $L(s, \hat{\chi}_f)$ satisfies, it is either

- if f is not a perfect square, a polynomial of degree $n-1$ in q^{-s}, or

- if f is a perfect square, then
$$L(s,\hat{\chi}_f)=\mathcal{Q}_f(s)\zeta(s)$$
where $\mathcal{Q}_f(s)$ is a polynomial in q^{-s} of degree n with $\mathcal{Q}_f(1)=1$.

Therefore, for f a nonsquare of degree less than or equal to m, we know that the m^{th} coefficient of $L(s,\hat{\chi}_f)$ vanishes, i.e.,
$$\sum_{deg\ g=m} b(g,f)\chi_{f_0}(\hat{g})=0$$
if $\deg f\leq m$, unless f is a perfect square. We write
$$Z(s,w)=Z_0(s,w)+Z_0(w,s)-Z_1(s,w)$$
where
$$Z_0(s,w)=\sum_{m\geq n\geq 0}\frac{1}{q^{ns}q^{mw}}\sum_{\substack{deg\ f=n\\ deg\ g=m}} b(g,f)\chi_{f_0}(\hat{g})$$
and
$$Z_1(s,w)=\sum_{n\geq 0}\frac{1}{q^{ns}q^{nw}}\sum_{\substack{deg\ f=n\\ deg\ g=n}} b(g,f)\chi_{f_0}(\hat{g}).$$

The nice thing now is that in evaluating Z_0 we only have to worry about when f is a perfect square. In this case, the character is $\chi_f(g)$ is not present, and we have a stronger multiplicativity statement which translates into an Euler product for a closely related series.

More precisely, let
$$Y_0(s,w)=\sum_{\substack{f,g\ \text{monic}\\ f\ \text{a perfect square}}}\frac{b(g,f)}{|f|^w|g|^s}.$$
Then Y_0 has an Euler product, and using our knowledge of $b(P^k,P^l)$ we may compute
$$Y_0(s,w)=\frac{1-q^{1-s-2w}}{(1-q^{1-2w})(1-q^{1-s})(1-q^{2-2s-2w})}.$$

5.4.1. *Convolutions of rational functions.* Let $R_1(x,y)$ and $R_2(x,y)$ be two rational functions, regular at the origin
$$R_1(x,y)=\sum_{j,k\geq 0} b_1(j,k)x^jy^k$$
$$R_2(x,y)=\sum_{j,k\geq 0} b_2(j,k)x^jy^k.$$
Then we let let $R_1\star R_2$ denote the power series defined by
$$(R_1\star R_2)(x,y)=\sum_{j,k\geq 0} b_1(j,k)b_2(j,k)x^jy^k.$$

Then $(R_1\star R_2)(x,y)$ is again a rational function of x and y. Indeed, write $(R_1\star R_2)(x,y)=$
$$\int\int R_1(z_1,z_2)R_2\left(\frac{x}{z_1},\frac{y}{z_2}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$
and evaluate the integral by partial fractions. The integrals here are taken over small circles centered at the origin.

5.4.2. *Computing $Z(s,w)$ (concl.).* The rest is easy: Since $Z_0 = Y_0 \star K$, for

$$K(x,y) = \sum_{m \geq n \geq 0} x^n y^m = \frac{1}{(1-x)(1-xy)},$$

we may compute

$$Z_0(s,w) = \frac{1}{(1-q^{1-w})(1-q^{3-2s-2w})}.$$

By a similar argument, we find

$$Z_1(s,w) = \frac{1}{(1-q^{3-2s-2w})}.$$

Putting everything together, we arrive at

$$Z(s,w) = \frac{1-q^{2-s-w}}{(1-q^{1-s})(1-q^{1-w})(1-q^{3-2s-2w})}$$

or after setting $x = q^{-s}, y = q^{-w}$,

$$\boxed{Z(s,w) = \frac{1-q^2xy}{(1-qx)(1-qy)(1-q^3x^2y^2)}}$$

This computation was first done by a different method by Fisher and Friedberg, [**29**]. In [**29**] there also appears a higher genus example.

6. Concluding Remarks

We conclude by mentioning several additional applications of multiple Dirichlet series to automorphic forms and analytic number theory.

6.1. Unweighted multiple Dirichlet series. Most of this article has concerned perfect multiple Dirichlet series—functions that continue to the full product of complex planes. Such objects (when they exist) depend on summing L-series times weighting factors. It is natural to ask what would happen without the weight factors. In [**20**], Chinta, Friedberg and Hoffstein show that it is possible to continue *unweighted* multiple Dirichlet series and to thereby get information inside the critical strip. They obtain mean value results, including a mean value theorem for products of L-functions, inside the critical strip but successively farther from the center as the degree of the Euler product increases. They also obtain a distribution result for these L-functions at $s = 1$.

A consequence of their main theorem is the following non-vanishing theorem.

THEOREM 6.1. [**20**] *Fix $n \geq 2$. Let F be a global field containing n n-th roots of unity, and let π_j, $1 \leq j \leq k$, be cuspidal automorphic representations of $GL_{r_j}(\mathbb{A}_F)$. Let $r = \sum_{j=1}^{m} r_j$, and suppose that $s_0 \in \mathbb{C}$ satisfies $\Re(s_0) > 1 - 1/(r+1)$. Then there exist infinitely many characters χ of order exactly n such that*

$$L(s_0, \pi_i \otimes \chi) \neq 0 \qquad (1 \leq i \leq k).$$

If $n = 2$, the conclusion is true if $\Re(s_0) > 1 - 1/r$.

6.2. Relation of multiple Dirichlet series to predictions about moments arising from random matrix theory. In [**25**], Diaconu, Goldfeld, and Hoffstein applied the work [**15**] of Bump, Friedberg and Hoffstein on GL(3), described in Section 3.6 above, to Eisenstein series on GL(3) to obtain mean value results for cubes of quadratic L-series. The error term obtained improved on the recent results of Soundararajan [**53**]. Moreover, they showed that natural conjectures concerning the continuation of sums of quadratic twists of higher moments, though this analytic continuation is expected to have an essential boundary, could be used to derive conjectural formulas for arbitrary moments of the zeta function and of quadratic L-series. These formulas agree with those of Conrey, Farmer, Keating, Rubinstein, and Snaith [**21**], derived by random matrix methods.

6.3. Weyl group multiple Dirichlet series. One can attach a multiple Dirichlet series to an integer n and an arbitrary reduced root system, whose coefficients are products of n-th order Gauss sums. It is expected that the series constructed from higher twists described in Section 4 arise naturally as residues of these Weyl group multiple Dirichlet series. These series are described further in the paper [**7**] in this volume. We give here one example of the application of these series to number theory.

The quadratic ($n = 2$) multiple Dirichlet series associated to A_5 has the nice property that it is essentially a sum of zeta functions of biquadratic extensions of the base field. This multiple Dirichlet series is roughly of the form

$$\sum_{d_2,d_4} \frac{L(s_1,\chi_{d_2})L(s_3,\chi_{d_2d_4})L(s_5,\chi_{d_4})}{d_2^{s_2}d_4^{s_4}}.$$

Using the analytic continuation of this series Chinta [**18**] has established a mean value result for this product of L-functions. For example, when the base field is $\mathbb{Q}$, we have

THEOREM 6.2. [**18**]

$$\sum_{\substack{d_1,d_2>0\\ d_1d_2<x\\ odd}} a(d_1,d_2)L_2(\tfrac{1}{2},\chi_{d_1})L_2(\tfrac{1}{2},\chi_{d_2})L_2(\tfrac{1}{2},\chi_{d_1d_2})$$

$$\sim \frac{\zeta_2(\frac{3}{2})\zeta_2(2)^3}{48} X \log^4 X,$$

as $X \to \infty$.

The weighting factor $a(d_1,d_2)$ appearing in the theorem satisfies

- $a(d_1,d_2) = 1$ if d_1d_2 square-free
- The weights are "small" in the sense that, for d_1d_2 square-free,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2w}} \left(\sum_{m_1m_2=n^2} a(m_1d_1, m_2d_2) \right)$$

  converges absolutely for $\Re(w) > 1/2$.

An explicit description can be found in [**18**].

6.4. Over a number field, the theory of multiple Dirichlet series arising from a sum of twisted automorphic L-functions gives one a unified way to study many problems concerning growth in families of L-functions. Over a function field it gives rise to rational functions in several variables that are natural objects (and that one might wish to understand geometrically). In conclusion, it seems of genuine interest to develop the theory of multiple Dirichlet series further.

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