ASYMPTOTICS FOR SUMS OF TWISTED L-FUNCTIONS AND APPLICATIONS

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This paper is dedicated to Professor Stephen J. Rallis

Abstract. Let $n \geq 2$, let F be a global field containing a full set of n-th roots of unity, and let π be an isobaric automorphic representation of $GL_r(\mathbb{A}_F)$. We establish asymptotic estimates for the sum of the n-th order twisted L-functions of π, $L(s, \pi \otimes \chi)$, for s such that $\mathrm{Re}(s) > \max(1 - 1/r, 1/2)$ if $n = 2$ and $\mathrm{Re}(s) > 1 - 1/(r+1)$ if $n > 2$. As an application we establish new non-vanishing theorems for twists of given order, including a simultaneous nonvanishing result. When $n = 2$ and each factor of π is tempered we use this information on asymptotics to prove that the twisted L-values at $s = 1$ give rise to a distribution function.

1. Introduction

Let F be a global field and let π be a unitary isobaric automorphic representation of $GL_r(\mathbb{A}_F)$ with L-function $L(s, \pi)$. In this paper we study the family of twisted L-functions $L(s, \pi \otimes \chi)$, where χ ranges over the idèle class characters of *fixed* finite order. In particular we establish the existence of nonvanishing twists for given s inside the critical strip but sufficiently close to the edge.

The existence of nonvanishing twists of some finite order for π on $GL(2)$ was established by Rohrlich [Ro]. Building on this work, Barthel and Ramakrishnan [BR] show that for π a cuspidal automorphic representation of $GL_r(\mathbb{A}_F)$, $r \geq 3$, and given s with $\mathrm{Re}(s) > 1 - 1/r$ there exist infinitely many primitive ray class characters χ such that $L(s, \pi \otimes \chi) \neq 0$. Under the stronger assumption of temperedness, Barthel and Ramakrishnan find infinitely many nonvanishing twists for s satisfying $\mathrm{Re}(s) > 1 - 2/(r+1)$. Luo, Rudnick and Sarnak [LRS]

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show a similar result in the case of a degree r Rankin-Selberg convolution $L(s, \pi_1 \times \bar{\pi}_1)$ (so r is a square), replacing the assumption of temperedness by the observation that the Rankin-Selberg L-series has positive coefficients. By Langlands functoriality, this L-series is conjecturally of the form $L(s, \pi)$ for some isobaric automorphic representation π.

We sharpen these results by showing that, in fact, for any global field F and any isobaric automorphic representation π of $GL_r(\mathbb{A}_F)$, $r \geq 2$, for given s with $\mathrm{Re}(s) > 1 - 1/r$ there exist infinitely many *quadratic* characters χ such that $L(s, \pi \otimes \chi) \neq 0$. In addition, if the field F contains the n-th roots of unity for some $n > 2$, then for s with $\mathrm{Re}(s) > 1 - 1/(r+1)$ there exist infinitely many characters χ *of order exactly* n such that $L(s, \pi \otimes \chi) \neq 0$. In both these results the characters may be specified at a finite number of places.

The methods of Barthel-Ramakrishnan combined with the large sieve can also be used to give non-vanishing results for twists by characters of fixed order near $\mathrm{Re}(s) = 1$. However, the results of this paper—in particular, the uniformity of the interval of nonvanishing over all number fields—seem to be stronger than can be obtained by the large sieve in its present form. Our methods are based on properties of automorphic L-functions and double Dirichlet series.

Our results are formulated for a general isobaric automorphic representation. Let π_j, $1 \leq j \leq k$, be cuspidal automorphic representations of $GL_{r_j}(\mathbb{A}_F)$ with $r = \sum_{j=1}^m r_j$. Recall that Langlands's theory of Eisenstein series allows one to construct an isobaric automorphic representation $\pi = \boxplus_{j=1}^m \pi_j$ of $GL_r(\mathbb{A}_F)$ such that the standard L-function of π is given by

$$L(s, \pi) = \prod_{j=1}^{m} L(s, \pi_j). \tag{0.1}$$

Since $L(s, \pi \otimes \chi) = \prod_{j=1}^m L(s, \pi_j \otimes \chi)$, our nonvanishing result establishes a simultaneous nonvanishing theorem for twists.

These results on nonvanishing are consequences of our main theorem, which we now state.

Let $n \geq 2$ be a fixed integer, and let F be a global field with n n-th roots of unity. Let S be a finite set of places containing all archimedean places, all places dividing n, and such that the class number of the ring of S-integers is 1. To each square-free ideal d prime to S one may attach (as in Fisher-Friedberg [FF1]) an idèle class character χ_d of order n such that if $d = (d_0)$ is principal with d_0 sufficiently congruent to 1 then χ_d is the character attached by class field theory to the extension $F(\sqrt[n]{d_0})/F$. Let C be a sufficiently large ideal supported on S (see Section 2 for the precise condition) and let H_C be the (narrow) ray class group modulo C. For π an automorphic representation let $L_S(s, \pi)$ denote its standard L-function with the factors at $v \in S$ removed. Then we have:

Theorem 1.1. *Let π be a unitary isobaric automorphic representation of $GL_r(\mathbb{A}_F)$ (not necessarily cuspidal) which is unramified outside S. Let A be a class in the ray class group H_C. Fix $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > \max(1 - 1/r, \frac{1}{2})$ *if $n = 2$ and* $\mathrm{Re}(s) > 1 - 1/(r+1)$ *if $n > 2$.*

(1) *Let $k > 0$ be a sufficiently large integer, depending on F, n, r. Then for any $\epsilon > 0$*

$$\sum_{\substack{d \in A \\ d \text{ sq-free} \\ 1<|d|<X}} L_S(s, \pi \otimes \chi_d) \left(1 - \frac{|d|}{X}\right)^k = \frac{c_S(s)X}{k+1} + O_\epsilon(X^{\frac{1}{2}+\epsilon})$$

as $X \to \infty$, where the sum is over square-free integral ideals d in the ray class A, and where $c_S(s)$ is a constant given explicitly below (see equation (2.3)). If S is sufficiently large then $c_S(s)$ is nonzero.

(2) *Let $n = 2$, let π be self-contragredient, and let s be real. If $L_S(s, \pi \otimes \chi_d) \geq 0$ for all d then*

$$\sum_{\substack{d \in A \\ d \text{ sq-free} \\ 1<|d|<X}} L_S(s, \pi \otimes \chi_d) \sim c_S(s)X$$

as $X \to \infty$, where the sum is over square-free integral ideals d in the ray class A of absolute norm between 1 and X.

Similar results hold for the full L-function $L(s, \pi \otimes \chi_d)$ in place of $L_S(\pi \otimes \chi_d)$. See Section 3.

Corollary 1.2. *Let π be a unitary isobaric automorphic representation of $GL_r(\mathbb{A}_F)$, $r \geq 2$. Let A be a class in the ray class group H_C. Let $\mathrm{Re}(s) > 1 - 1/(r+1)$. Then there exist infinitely many characters χ_d, $d \in A$, of order n such that*

$$L(s, \pi \otimes \chi_d) \neq 0.$$

If $n = 2$, the conclusion is true for $\mathrm{Re}(s) > 1 - 1/r$.

Since Theorem 1.1 applies to all isobaric automorphic representations it also gives information about moments; see (0.1). In the case that $\pi = \boxplus_{j=1}^m \pi_j$ with π_j tempered, this information is sufficient to establish the existence of a distribution function $F(y)$ giving the limiting proportion of the quadratically twisted L-values $L(1, \pi \otimes \chi)$ of magnitude at most y. For the simplest case, that is, for $L(s, \chi_d)$ the family of Dirichlet L-series associated to the Kronecker symbols χ_d, such a distribution result was obtained (in fact, for $d > 0$ and $\mathrm{Re}(s) > 3/4$) by Chowla and Erdös [CE]. The analogue of Theorem 1.1 was established by Barban [Ba], who showed that for $k \in \mathbb{N}$,

$$\sum_{1<-d<X} L(1, \chi_d)^k \sim r_k X,$$

where the r_k are explicitly given non-zero constants. This asymptotic estimate for the k-th moments together with a theorem of Fréchet and Shohat [FS] allows one to establish a Chowla-Erdös type value distribution result for $d < 0$ and $s = 1$; see [Ba] and Section 3 below.

In Section 3 we shall use a similar method to show

Corollary 1.3. *Suppose that $\pi = \boxplus_{j=1}^m \pi_j$ is a unitary isobaric automorphic representation with each π_j tempered. Let $n = 2$. Fix S, A as above. Then there exists a non-decreasing right-continuous function $F_{\pi,A} : \mathbb{R} \to [0,1]$ such that*

$$\lim_{X\to\infty} \frac{|\{d \in A, d \text{ sq-free}, |d| < X\colon |L(1,\pi\otimes\chi_d)| < y\}|}{|\{d \in A, d \text{ sq-free}, |d| < X\}|} \to F_{\pi,A}(y)$$

at all points of continuity y of $F_{\pi,A}(y)$.

The requirement of temperedness here is necessary to fulfill the hypotheses for the theorem of Fréchet and Shohat.

Notice that if L is a motivic L-function of odd degree then 1 is a critical value in the sense of Deligne and these L-values are of arithmetic interest. For example,

Corollary 1.4. *Let k be a totally real number field. Let A be the trivial ray class in H_C. Then there exists a non-decreasing right-continuous function $F_k : \mathbb{R} \to [0,1]$ such that*

$$\lim_{X\to\infty} \frac{|\{(d) \in A, (d) \text{ sq-free}, d \gg 0, |d| < X\colon \frac{h(k(\sqrt{-d}))}{w_d Q_d \sqrt{f_d}} < y\}|}{|\{d \in A, d \text{ sq-free}, |d| < X\}|} \to F_k(y)$$

for all points of continuity y of $F_k(y)$. Here $h(k(\sqrt{-d}))$ is the relative class number of $k(\sqrt{-d})/k$, w_d is the number of roots of unity in $k(\sqrt{-d})$, f_d is the norm of the conductor of χ_{-d} and $Q_d = 1$ or 2 as in [Wa]*, Theorem 4.12.*

This follows from the Dirichlet class number formula for $k(\sqrt{-d})/k$ (see [Wa], for example). Similarly, one may adjoin $\sqrt{d}$ where $d \in k$ has exactly one positive embedding . A moment result in such a case was proved with certain restrictions on k by Peter in [Pe].

Distribution results similar to Cors. 1.3 and 1.4 can also be obtained by the use of the large sieve as noted above; we thank Peter Sarnak for pointing this out to us. Also, in the case of the Dirichlet L-series $L(s,\chi_d)$, much more is known about the distribution function, see for example [GS] and the references there.

The proofs of our results are based on a double Dirichlet series obtained by summing the twisted L-functions. Such series first arose from the study of certain Hecke-Rankin-Selberg type integrals of metaplectic Eisenstein series (see [BFH1]), but it is not apparent that the series used here may be so-obtained. Instead, our work is based on the convexity principle for holomorphic functions of two complex variables, whose use to study such series was first observed by Bump, Friedberg and Hoffstein [BFH2]. In [DGH,FF1-2] a similar point of view was taken. In particular, in [DGH] the approach of searching for a multiple Dirichlet series which is a weighted sum of twisted L-functions with a full meromorphic continuation in all variables was formalized. Unfortunately the desired weighting factors are not easy to obtain (see [BFH,FF1-2] for some low rank instances when $n = 2$). In this paper we take an opposite point of view - studying a multiple Dirichlet series which is an unweighted sum of twisted L-functions. This object has more limited meromorphic continuation but we show that this continuation still has consequences. We analyze the multiple Dirichlet series in several ways (one related to metaplectic Eisenstein series), and use this analysis to establish its continuation to different overlapping tube domains. By the convexity principle the double Dirichlet series then continues to the convex hull of the union of these regions. This information allows us to establish Theorem 1.1 by Tauberian methods.

2. The Double Dirichlet Series

Fix $n \geq 2$ and let F be a global field with n n-th roots of unity. In the number field case, let $\mathcal{O}$ denote the ring of integers of F, and let S_f be a finite set of non-archimedean places containing all places dividing n and such that the ring of S_f-integers $\mathcal{O}_{S_f}$ has class number 1. Let S_∞ denote the set of archimedean places and let $S = S_f \cup S_\infty$. Similarly, in the function field case, let $\mathcal{C}$ denote the smooth projective curve with function field F and choose a finite set of places S such that the divisors supported on S represent all classes in $\mathrm{Pic}(\mathcal{C})$. Let $\left(\frac{a}{*}\right)$ be the power residue symbol attached to the extension $F(\sqrt[n]{a})$ of F. To properly formulate the double Dirichlet series, let us extend the n-th power residue symbol as in Fisher-Friedberg [FF1]. We review the definition.

For each place v, let F_v denote the completion of F at v. Suppose first that F is a number field. For v nonarchimedean, let P_v denote the corresponding ideal of $\mathcal{O}$. Let $C = \prod_{v \in S_f} P_v^{n_v}$ with $n_v \geq 1$ sufficiently large that $a \in F_v$, $\mathrm{ord}_v(a-1) \geq n_v$ implies that $a \in (F_v^\times)^n$. Let H_C be the ray class group modulo C (the narrow ray class group if $n = 2$ and F has real embeddings), and let $R_C = H_C \otimes \mathbb{Z}/n\mathbb{Z}$. Write the finite group R_C as a direct product of cyclic groups, choose a generator for each, and let $\mathcal{E}_0$ be a set of ideals of $\mathcal{O}$ prime to S which represent these generators. For each $E_0 \in \mathcal{E}_0$ choose $m_{E_0} \in F^\times$ such that $E_0 \mathcal{O}_{S_f} = m_{E_0} \mathcal{O}_{S_f}$. Let $\mathcal{E}$ be a full set of representatives for R_C of the form $\prod_{E_0 \in \mathcal{E}_0} E_0^{n_{E_0}}$, $n_{E_0} \in \mathbb{Z}$. If $E = \prod_{E_0 \in \mathcal{E}_0} E_0^{n_{E_0}}$ is such a representative, then let $m_E = \prod_{E_0 \in \mathcal{E}_0} m_{E_0}^{n_{E_0}}$. Note that $E\mathcal{O}_{S_f} = m_E \mathcal{O}_{S_f}$ for all $E \in \mathcal{E}$. For convenience only we suppose that $\mathcal{O} \in \mathcal{E}$ and $m_{\mathcal{O}} = 1$.

Let $J(S)$ denote the group of fractional ideals of $\mathcal{O}$ coprime to S_f. Let $I, I_1 \in J(S)$ be coprime. Write $I = (m)EG^n$ with $E \in \mathcal{E}$, $m \in F^\times$, $m \equiv 1 \bmod C$, m positive under all real embeddings of F (we write $m \gg 0$; this condition is trivial unless $n = 2$) and $G \in J(S)$, $(G, I_1) = 1$. Then as in [FF1], the n-th power residue symbol $\left(\frac{m m_E}{I_1}\right)$ is defined, and if $I = (m')E'G'^n$ is another such decomposition, then $E' = E$ and $\left(\frac{m' m_E}{I_1}\right) = \left(\frac{m m_E}{I_1}\right)$.

In view of this define the n-th power residue symbol $\left(\frac{I}{I_1}\right)$ by $\left(\frac{I}{I_1}\right) = \left(\frac{m m_E}{I_1}\right)$ and the character χ_I by $\chi_I(I_1) = \left(\frac{I}{I_1}\right)$. This depends on the choices above, but we suppress this from the notation. Let S_I denote the support of the conductor of χ_I. One may check that if $I = I'G^n$, then $\chi_I(I_1) = \chi_{I'}(I_1)$ whenever both are defined. This allows one to extend χ_I to a character of all ideals of $J(S \cup S_I)$. One has, as in [FF1]

Proposition 2.1. Reciprocity – *Let I, $J \in J(S)$ be coprime, and $\alpha(I, J) = \chi_I(J)\chi_J(I)^{-1}$. Then $\alpha(I, J)$ depends only on the images of I and J in R_C.*

The definitions in the function field case are similar, but it is move traditional to work with divisors and write them additively. For more details see [FF1-2], where the case $n = 2$

is discussed in detail. Throughout this section we keep the number field notation, but the argument for function fields is the same.

For d an integral ideal, let $|d|$ denote its absolute norm. Let $\mathcal{I}(S)$ denote the integral ideals prime to S_f. Let π be an isobaric automorphic representation on $GL_r(\mathbb{A}_F)$, not necessarily cuspidal, which is unramified outside S and let $L_S(s, \pi \otimes \chi_d)$ be the L-function for π twisted by the character χ_d, with the primes in S removed. (Note that the Euler factor is also 1 at the primes dividing d.)

Before proceeding to give the proof of Theorem 1.1, let us record a necessary observation about the functional equation of such a π. So let π be as above. Then there is a partition $r = \sum_{j=1}^{m} r_j$ and a set of unitary cuspidal automorphic representations π_j such that $\pi \cong \oplus_{j=1}^{m} \pi_j$. If ξ is an idéle class character then the twisted L-function $L(s, \pi \otimes \xi)$ satisfies a functional equation

$$L(s, \pi \otimes \xi) = \epsilon(s, \pi \otimes \xi)\, L(1 - s, \tilde{\pi} \otimes \xi^{-1}), \tag{2.1}$$

where $\epsilon(s, \pi \otimes \xi)$ is the epsilon factor of $\pi \otimes \xi$.

Lemma 2.2. *Let $d, e \in \mathcal{O}(S)$ be square-free, and suppose that $\chi_d = \chi_e \chi_m$ with $m \in F^\times$, $m \equiv 1 \bmod C$ and $m \gg 0$ if $n = 2$ (this equality holds if and only if d, e project to the same class in R_C). Then $\epsilon(s, \pi \otimes \chi_d) = \epsilon(1/2, \chi_m)^r\, \chi_\pi(d/e)|d/e|^{r(1/2-s)}\epsilon(s, \pi \otimes \chi_e)$.*

Here $\epsilon(1/2, \chi_m)$ is given by a (normalized) n-th order Gauss sum, as in Tate's thesis. If $n = 2$ then it is identically 1.

Proof. We have $\epsilon(s, \pi) = |D^r f(\pi)|^{1/2-s}\epsilon(1/2, \pi)$ where D is the different of F, $f(\pi)$ is the conductor of π, and $\epsilon(1/2, \pi)$ is the central value of the epsilon-factor of π. Since π is unramified outside S, this central value is a product of local factors for $v \in S$ (each depending also on the choice of additive character). Now $\chi_d = \chi_e \chi_m$ where $m \equiv 1 \bmod C$. By the hypotheses on C, this implies that the local character $\chi_{m,v} = 1$ for all $v \in S$, and the local factors for $v \in S$ are equal. Since we are working with the twisted representations $\pi \otimes \chi_d$, $\pi \otimes \chi_e$, we must also compute the contributions from the places v dividing d or e. At these places each $\pi_{j,v}$ is unramified principal series and the comparison is obtained from the comparison of GL_1-epsilon-factors. These are computed as in Tate's thesis. (For more details see Lemma 2.2 in [FF1] and the proof of Theorem 3.1 in [FF2].) ∎

Proof of Theorem 1.1. We turn to the double Dirichlet series construction. The L-function for π is of the form

$$L(s, \pi) = L_S(s, \pi)\, L(s, \pi; S)$$

where $L(s, \pi; S)$ is the contribution from places $v \in S$ and where the contribution from the places prime to S may be written as a Dirichlet series

$$L_S(s, \pi) = \sum_{m \in \mathcal{I}(S)} a(m)\, |m|^{-s}$$

(m ranging over the ideals of $\mathcal{O}$ prime to S). Let A be a class in H_C. Let

$$Z_A(s,w;\pi) = \sum_{d\in A}{}' \frac{L_S(s,\pi\otimes\chi_d)}{|d|^w},$$

where the sum is over integral ideals d in A, and the prime on summation indicates d square-free. Since $L(s,\pi)$ converges absolutely for $\mathrm{Re}(s)>1$, the double Dirichlet series $Z_A(s,w;\pi)$ is absolutely convergent for $\mathrm{Re}(s)>1, \mathrm{Re}(w)>1$. We are interested in obtaining the analytic continuation of $Z_A(s,w;\pi)$ to a region containing $\mathrm{Re}(s)=1, \mathrm{Re}(w)=1$.

On the one hand, we have the convexity bound for $L(s,\pi\otimes\chi_d)$ which follows from the functional equation (2.1). If $L(s,\pi\otimes\chi_d)$ is entire then

$$L_S(s,\pi\otimes\chi_d) \ll_\epsilon \max\{1, |d|^{r(1-\sigma)/2+\epsilon}, |d|^{r(1/2-\sigma)+\epsilon}\},$$

for $\epsilon>0$, where $\sigma=\mathrm{Re}(s)$. If ℓ is the maximum order pole of $L(s,\pi\otimes\chi_d)$ for $d\in A$ then we conclude that $(s-1)^\ell\, Z_A(s,w;\pi)$ is absolutely convergent for

$$\mathrm{Re}(w) > \max\{1, \frac{r}{2}(1-\mathrm{Re}(s))+1, r(\frac{1}{2}-\mathrm{Re}(s))+1\}.$$

To extend past the domain of absolute convergence let us interchange the order of summation. We obtain

$$\begin{aligned} Z_A(s,w;\pi) &= \sum_{d\in A}{}' \frac{L_S(s,\pi\otimes\chi_d)}{|d|^w} \\ &= \sum_{m\in\mathcal{I}(S)} \frac{a_m}{|m|^s} \sum_{d\in A,\ (d,m)=1}{}' \frac{\chi_d(m)}{|d|^w}. \end{aligned}$$

For $m\in\mathcal{I}(S)$, let $[m]$ denote the class of m in H_C. Also let h_C be the order of the ray class group H_C and $\hat{H}_C$ be its group of characters. Upon using the power reciprocity law as formulated in Proposition 2.1 and the orthogonality relations, for $m\in\mathcal{I}(S)$ we obtain

$$\begin{aligned} \sum_{d\in A,(d,m)=1}{}' \frac{\chi_d(m)}{|d|^w} &= \alpha(A,[m]) \sum_{d\in A,(d,m)=1}{}' \frac{\chi_m(d)}{|d|^w} \\ &= \frac{\alpha(A,[m])}{h_C} \sum_{\eta\in\hat{H}_C} \eta^{-1}(A)\, D_m(w,\eta\chi_m), \end{aligned}$$

where

$$D_m(w,\eta\chi_m) = \sum_{d\in\mathcal{I}(S),(d,m)=1}{}' \frac{\eta(d)\chi_m(d)}{|d|^w}.$$

If μ now denotes the Möbius function of ideals of $\mathcal{O}$, then

$$
\begin{aligned}
D_m(w,\eta\chi_m) &= \sum_{d\in\mathcal{I}(S),(d,m)=1}\ \sum_{e^2|d}\mu(e)\frac{\eta(d)\chi_m(d)}{|d|^w} \\
&= \sum_{e\in\mathcal{I}(S),(e,m)=1}\frac{\mu(e)\,\eta^2(e)\,\chi_m^2(e)}{|e|^{2w}}\sum_{d\in\mathcal{I}(S),(d,m)=1}\frac{\eta(d)\chi_m(d)}{|d|^w} \\
&= \frac{L_{m\cup S}(w,\eta\chi_m)}{L_{m\cup S}(2w,\eta^2\chi_m^2)}.
\end{aligned}
\tag{2.2}
$$

where the subscripts indicate that the primes in S and those dividing m are removed from the last quotient of GL_1 L-functions. Thus $D_m(w,\eta\chi_m)$ is holomorphic for $\mathrm{Re}(w)\geq 1/2$ if $\eta\chi_m$ is not the trivial character; otherwise, it is meromorphic in this region with only a simple pole at $w=1$. Note that this occurs only when $\eta=1$ and m is a perfect n-th power.

It is the residue at this pole which gives the main term in Theorem 1.1. Note that for $\mathrm{Re}(s)>1$,

$$
c_S(s)=\mathrm{Res}_{w=1}\,Z_A(s,w;\pi)=\frac{\kappa}{\zeta_F(2)\,h_C}\left(\prod_{v\in S}\frac{\zeta_{F,v}(2)}{\zeta_{F,v}(1)}\right)\sum_{m\in\mathcal{I}(S)}\frac{a_{m^n}}{|m|^{ns}}\prod_{v|m}\frac{\zeta_{F,v}(2)}{\zeta_{F,v}(1)},
\tag{2.3}
$$

where $\zeta_F=\prod\zeta_{F,v}$ is the Dedekind zeta function of F and κ is its residue at $w=1$. We show later in this proof that this Dirichlet series representation, initially valid for $\mathrm{Re}(s)>1$, remains true for $\mathrm{Re}(s)>\max(1-1/r,\frac{1}{2})$ if $n=2$ and for $\mathrm{Re}(s)>1-1/(r+1)$ if $n>2$. Moreover, we will show that for S sufficiently large, the residue is non-zero for s in this range.

Next let us note that for $\epsilon>0,\mathrm{Re}(w)\geq 1/2, w\neq 1$ and $\mathrm{Im}\,w=t$,

$$
\sum_{m\in\mathcal{I}(S),|m|<X}\left|D_m(w,\eta\chi_m)\right|^2\ll_{C,\epsilon}X^{1+\epsilon}(1+|t|)^K.
\tag{2.4}
$$

Here K is a constant depending on the ground field F, and on n,r. Indeed, this follows from the evaluation (2.2) since the $L(w,\eta\chi_m)$ occur as Fourier coefficients of the Eisenstein series on the n-fold cover of GL_n induced from the theta function on the n-fold cover of GL_{n-1} (see Suzuki [Su] and Banks, Bump, and Lieman [BBL]); alternatively it follows by using the associated double Dirichlet series as in Diaconu [Di]. See also [FHL].

Moreover, from the properties of the Rankin-Selberg L-function $L(s,\pi\otimes\tilde{\pi})$, it follows that

$$
\sum_{|m|<X}|a_m|^2\ll X(\log X)^\lambda
$$

for some integer λ. Therefore by the Cauchy-Schwarz inequality,

$$
\sum_{|m|<X}\left|a_mD_m(w,\eta\chi_m)\right|^2\ll_{A,\epsilon}X^{1+\epsilon}(1+|t|)^K.
\tag{2.5}
$$

By partial summation, we conclude that $(w-1)Z_A(s,w;\pi)$ is holomorphic for $\mathrm{Re}(s)>1$, $\mathrm{Re}(w)>1/2$ and that

$$(w-1)Z_A(s,w;\pi) \ll_{A,\epsilon} (1+|t|)^K.$$

Combining this with the region of absolute convergence deduced from convexity estimates on $L_S(s,\pi\otimes\chi_d)$, we have established the holomorphicity of $(w-1)(s-1)^\ell Z_A(s,w;\pi)$ on the union of the tube domains

$$T_1 = \{(s,w) : \mathrm{Re}(w) > 1/2, \mathrm{Re}(s) > 1\}$$

and

$$T_2 = \{(s,w) : \mathrm{Re}(w) > \max(1, \tfrac{r}{2}(1-\mathrm{Re}(s)) + 1, r(\tfrac{1}{2} - \mathrm{Re}(s)) + 1)\}.$$

In the case $n=2$ we can further enlarge the region of continuation as follows. For convenience let us suppose that $r \geq 2$; the case $r=1$ is similar but simpler. We apply the functional equation (2.1). (See also Remark 2.5 below.) Note that for $d \in A$ the product of local L-factors $L(s,\pi\otimes\chi_d;S)$ depends only on A; so we write it $L(s,\pi\otimes\chi_A;S)$. Also, by Lemma 2.2 we may write $\epsilon(s,\pi\otimes\chi_d) = A(s)\,\chi_\pi(d)|d|^{r(s-1/2)}$ where $A(s)$ is a monomial function depending only on π and A. Thus we have

$$\begin{aligned} Z_A(s,w;\pi) &= \sum_{d\in A}{}' \frac{L_S(s,\pi\otimes\chi_d)}{|d|^w} \\ &= \frac{L(1-s,\tilde\pi\otimes\chi_A;S)}{L(s,\pi\otimes\chi_A;S)} \sum_{d\in A}{}' \frac{\chi_\pi(d)\,L_S(1-s,\tilde\pi\otimes\chi_d)}{|d|^{w+r(s-1/2)}}. \end{aligned}$$

By interchanging summation and using the Cauchy-Schwarz inequality as above, one thus sees that $Z_A(s,w;\pi)$ has holomorphic continuation to the tube domain

$$T_3 := \{(s,w) : \mathrm{Re}(s) < 0, \mathrm{Re}(w) > -r\,\mathrm{Re}(s) + r/2 + 1/2\}.$$

By the analytic continuation of holomorphic functions of two complex variables on tube domains to the convex hull [Hö], the function $(s-1)^\ell(w-1)Z_A(s,w;\pi)$ thus has holomorphic continuation to the convex hull of the tube domain $T_1 \cup T_2$ for $n \geq 3$ (resp. $T_1 \cup T_2 \cup T_3$ for $n=2$).

Note that $L_S(s,\pi\otimes\chi_d)$ is entire for $d\in A$ unless $d=(1)$. Hence $Z_A(s,w;\pi)$ can have a pole at $s=1$ only if $(1)\in A$. If this is the case, we will find it more convenient to work with $Z_A(s,w;\pi) - L_S(s,\pi)$. Let us define

$$Z_A^0(s,w;\pi) = \begin{cases} Z_A(s,w;\pi) - L_S(s,\pi) & \text{if } (1)\in A \\ Z_A(s,w;\pi) & \text{otherwise.} \end{cases}$$

Observe that

$$\mathrm{Res}_{w=1}\, Z_A^0(s,w;\pi) = \mathrm{Res}_{w=1}\, Z_A(s,w;\pi).$$

Theorem 1.1 will now be a consequence of a standard Tauberian argument, as follows. First observe that as the convex hull of the tube domains is a domain of holomorphy it follows from the Phragmen-Lindelöf principle that the function $(w-1)Z_A^0(s,w;\pi) \ll_{A,\epsilon} (1+|t|)^K$ in the entire domain. Choose k to be a positive integer with $k > K+1$, with K as in (2.4). Applying the integral transform

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^w\,dw}{w(w+1)\dots(w+k)} = \begin{cases} \frac{1}{k!}(1-1/x)^k & \text{if } x>1 \\ 0 & \text{if } 0<x\le 1 \end{cases}$$

we obtain first

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{Z_A^0(s,w;\pi)\,x^w\,dw}{w(w+1)\dots(w+k)} = \frac{1}{k!}\sum_{\substack{d\in A, d\neq(1)\\ d \text{ sq-free}}} L_S(s,\pi\otimes\chi_d)\left(1-\frac{|d|}{x}\right)^k.$$

Moving the line of integration to $\operatorname{Re} w = 1/2+\epsilon$, for $\epsilon>0$, the above equals

$$\frac{1}{(k+1)!}\operatorname{Res}_{w=1} Z_A^0(s,w;\pi)x + \frac{1}{2\pi i}\int_{1/2+\epsilon-i\infty}^{1/2+\epsilon+i\infty} \frac{Z_A^0(s,w;\pi)\,x^w\,dw}{w(w+1)\dots(w+k)},$$

as $Z_A^0(s,w;\pi)$ is an analytic function of w in this region except for a simple pole at $w=1$. The integral converges absolutely and is bounded above by $x^{1/2+\epsilon}$, so letting $c_S(s) = \operatorname{Res}_{w=1} Z_A(s,w;\pi)$ we obtain part (1) of Theorem 1.1. In the case of part (2) the values being averaged are non-negative, so the Wiener–Ikehara Tauberian theorem applies to give an unweighted asymptotic result.

To complete the proof of Theorem 1.1 we must now analyze the residue of $Z_A(s,w;\pi)$ at $w=1$. This is given for $\operatorname{Re}(s)>1$ by the expression (2.3). Thus this residue is a nonzero constant times the Euler product

$$\prod_{(v,S)=1}\left(1+\frac{q_v}{q_v+1}\sum_{g=1}^{\infty}\frac{a_{P_v^{ng}}}{q_v^{ngs}}\right) \tag{2.6}$$

where we have set $|P_v| = q_v$ for convenience. We analyze the convergence of this product separately in the cases $n>2$ and $n=2$.

Suppose that the Satake parameters of π_v are given by $\alpha_{v,i}$, $1\le i\le r$. Then by the result of Luo, Rudnick and Sarnak [LRS], $|\alpha_{v,i}| \le q_v^{1/2-1/(r^2+1)}$. Thus

$$|a_{P_v^{ng}}| = \sum_{k_1+\dots+k_r=ng}\prod_i \alpha_{v,i}^{k_i} \le p_r(ng)q_v^{ng(1/2-1/(r^2+1))},$$

where $p_r(ng)$ is the number of ordered partitions of ng into r non-negative integral pieces. If $r=2$ one may do better using the estimate of Kim and Shahidi [KS], namely $|\alpha_{v,i}| < q_v^{1/9}$. For later use, define

$$\delta = \delta_{r,n} = \begin{cases} 1/2 - 1/(r+1) + 1/(r^2+1) & \text{if } r>2, n>2 \\ 1/2 - 1/r + 1/(r^2+1) & \text{if } r>2, n=2 \\ 7/18 & \text{if } r=2. \end{cases}$$

Also, note the trivial bound

$$p_r(ng) \le (ng+1)^r.$$

First, suppose $n \ge 3$. Recall that an infinite product $\prod_v (1+b_v)$ converges absolutely if and only if the sum $\sum_v |b_v|$ converges. For $\mathrm{Re}(s) > 1 - 1/(r+1)$ we have

$$\begin{aligned}\sum_{(v,S)=1} \sum_{g=1}^{\infty} \frac{|a_{P_v^{ng}}|}{q_v^{ng\,\mathrm{Re}(s)}} &\le \sum_{(v,S)=1} \sum_{g=1}^{\infty} \frac{p_r(ng)}{q_v^{ng\delta}} \\ &\le \sum_{(v,S)=1} \sum_{g=1}^{\infty} \frac{(ng+1)^r}{q_v^{ng\delta}}.\end{aligned}$$

This series converges absolutely because $n\delta > 1$.

Now suppose instead that $n = 2$. To prove the absolute convergence of the Euler product (2.6), we make the observation that the residue is well approximated by the symmetric square L-function, which converges absolutely for $\mathrm{Re}(s) > 1$ by Rankin-Selberg theory. Note that if $\pi = \boxplus_{j=1}^m \pi_j$ then

$$L(s,\pi,\mathrm{Sym}^2) = \prod_{j=1}^{m} L(s,\pi_j,\mathrm{Sym}^2) \prod_{1\le j<j'\le m} L(s,\pi_j \times \pi_{j'})$$

so we make use of the analytic properties of Rankin-Selberg convolutions as well. Let

$$L(s,\pi,\mathrm{Sym}^2) = \prod_v T_v(s),$$

where

$$T_v(s) = \prod_{1\le i\le j\le r} (1-\alpha_{v,i}\alpha_{v,j} q_v^{-s})^{-1} = \prod_{1\le i\le r'} (1-\beta_{v,i} q_v^{-s})^{-1},$$

say, with $r' = r(r+1)/2$ and $|\beta_{v,i}| \le q_v^{1-2/(r^2+1)}$. Then for $\mathrm{Re}(s) > 1-1/r$,

$$\begin{aligned}T_v(2s) &= 1 + \frac{a(P_v^2)}{q_v^{2s}} + \sum_{g\ge 2} \frac{1}{q_v^{2gs}} \sum_{k_1+\cdots k_{r'}=g} \beta_{v,1}^{k_1}\cdots\beta_{v,r'}^{k_{r'}} \\ &= 1 + \frac{a(P_v^2)}{q_v^{2s}} + O_r(q_v^{-4\delta}).\end{aligned}$$

Similarly,

$$\begin{aligned}1 + \frac{q_v}{q_v+1}\sum_{g=1}^{\infty} \frac{a_{P_v^{2g}}}{q_v^{2gs}} &= 1 + \left(1 + O\left(\frac{1}{q_v}\right)\right)\left(\frac{a(P_v^2)}{q_v^{2s}} + \sum_{g\ge 2} \frac{a_{P_v^{2g}}}{q_v^{2gs}}\right) \\ &= 1 + \frac{a(P_v^2)}{q_v^{2s}} + O_r(q_v^{-1-2\delta}).\end{aligned}$$

Hence the quotient

$$\frac{1+\frac{q_v}{q_v+1}\sum_{g=1}^{\infty}\frac{a_{P_v^{2g}}}{q_v^{2gs}}}{T_v(2s)} = 1 + O_r(\max(q_v^{-4\delta}, q_v^{-1-2\delta})),$$

and the product over all places

$$\prod_v \left(1 + O_r(\max(q_v^{-4\delta}, q_v^{-1-2\delta}))\right)$$

converges absolutely. From the absolute convergence of the symmetric square L-function for $\mathrm{Re}(s) > 1$ the convergence of (2.6) follows.

In all cases the Euler product (2.6) is absolutely convergent for $\mathrm{Re}(s)$ as in Theorem 1.1. It is hence non-zero provided none of the terms vanishes. We may guarantee that none of the terms vanishes by throwing out a finite number of primes, i.e. by enlarging the set S. Explicitly, we desire q_v sufficiently large that

$$\frac{q_v}{q_v+1}\sum_{g=1}^{\infty}\frac{a_{P_v^{ng}}}{q_v^{ngs}}$$

is less than 1 in absolute value, for $\mathrm{Re}(s) > 1 - 1/r$ if $n = 2$ or for $\mathrm{Re}(s) > 1 - 1/(r+1)$ if $n \geq 3$. Using the absolute bound for the Fourier coefficients, for such s one has

$$\left|\frac{q_v}{q_v+1}\sum_{g=1}^{\infty}\frac{a_{P_v^{ng}}}{q_v^{ngs}}\right| \leq \sum_{g=1}^{\infty}\frac{p_r(ng)}{q_v^{ng\delta}}.$$

For the partition function

$$p_r(ng) = \binom{ng+r-1}{r-1} = \frac{(ng+r-1)\cdots(ng+1)}{(r-1)!}$$

we use the bounds

$$p_r(ng) \leq \frac{(2r)^{r-1}}{(r-1)!},$$

for $ng \leq r$, and by Stirling's formula

$$p_r(ng) \leq A^{ng}$$

if $ng > r$, for some absolute constant A. Therefore, if q_v is sufficiently large,

$$\begin{aligned}\sum_{g=1}^{\infty}\frac{p_r(ng)}{q_v^{ng\delta}} &= \sum_{g=1}^{[r/n]} + \sum_{g\geq r/n} \\ &\leq \frac{(2r)^{r-1}}{(r-1)!}\sum_{g=1}^{[r/n]} q_v^{-ng\delta} + \sum_{g\geq r/n}(A/q_v^{\delta})^{ng} \\ &\leq C_1\frac{(2r)^{r-1}}{(r-1)!}q_v^{-n\delta} + C_2(A/q_v^{\delta})^{r}\end{aligned}$$

for some absolute constants C_1, C_2. This quantity will be less than 1 provided

$$q_v^{n\delta} >> \max\left\{\frac{(2r)^{r-1}}{(r-1)!}, A^n\right\}.$$

This concludes the proof of Theorem 1.1. ∎

Remark 2.3. Changing the choices made in defining the characters χ_d above would amount to changing π to $\pi\otimes\chi$ where χ is an idèle class character which is obtained from a character of R_C.

Remark 2.4. In fact, the proof of Theorem 1.1 does not make use of the full hypothesis that π is an automorphic representation of $GL_r(\mathbb{A}_F)$. If $D(s)$ is any Eulerian Dirichlet series, absolutely convergent for $\mathrm{Re}(s) > 1$, with suitable functional equations under GL_1 twists, satisfying an estimate for the growth of the coefficients as in (2.5), and such that the residue (2.6) is well-behaved, then the proof of Theorem 1.1 goes through and the growth estimates of Theorem 1.1 hold. For example, this holds for any Dirichlet series in the Selberg class which has suitable functional equations under GL_1 twists.

Remark 2.5. One may also make use of the functional equation to increase the region of continuation of $Z_A(s, w; \pi)$ in the cases $r = 1$, n arbitrary and $r = 2$, $n = 3$. We shall not pursue these cases here since in fact further work allows one to continue an appropriate modified double Dirichlet series to all of $\mathbb{C}^2$. (See Friedberg-Hoffstein-Lieman [FHL] for the first case and Brubaker-Bump-Friedberg-Hoffstein [BBFH] for the second.)

3. Value distribution of $L(1, \pi \otimes \chi_d)$

Let $\{\mu_n\}$ be a sequence of Borel probability measures on $\mathbb{R}$. One says that this sequence converges weakly to a measure μ if

$$\int f d\mu_n \to \int f d\mu, \text{ as } n \to \infty$$

for all continuous, compactly supported functions f on $\mathbb{R}$. Suppose F (resp. F_n) is the distribution function of μ (resp. μ_n), i.e., F is the non-decreasing right-continuous function defined by

$$F(x) = \mu((-\infty, x]).$$

Then $\mu_n \to \mu$ weakly if and only if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all points of continuity x of F. The k-th moment of a measure μ on $\mathbb{R}$ is defined to be

$$M_k[\mu] := \int_{\mathbb{R}} x^k d\mu(x) = \int x^k dF(x)$$

if this limit exists.

A theorem of Fréchet and Shohat [FS],[Bi] allows us to deduce weak convergence from convergence of moments. Precisely: let $\{\mu_n\}$ be a sequence of Borel probability measures on $\mathbb{R}$ such that for every positive integer k, the limit

$$M_k := \lim_{n\to\infty} M_k[\mu_n]$$

exists and the power series

$$\Psi(w) := \sum_{k\geq 0} \frac{M_k w^k}{k!}$$

has positive radius of convergence. Then there exists a measure μ such that

(i) μ_n converges weakly to μ,
(ii) for all positive integers k, $M_k = M_k[\mu]$, and
(iii) for $t \in \mathbb{R}$ sufficiently small, the characteristic function ψ of μ,

$$\psi(t) := \int_{\mathbb{R}} e^{itz} d\mu z,$$

is given by

$$\Psi(it) = \psi(t).$$

We now turn to the proof of Corollary 1.3. For a positive integer X, we define the probability measure μ_X (depending on A) by

$$\mu_X([0,y)) := \frac{|\{d \in A, d \text{ sq-free}, |d| < X \colon |L(1, \pi\otimes\chi_d)|^2 < y\}|}{|\{d \in A, d \text{ sq-free}, |d| < X\}|}$$

for all $y > 0$. Fix S as in Theorem 1.1. Let $n = 2$. Since for $d \in A$ the local character $\chi_{d,v}$ is independent of d for $v \in S$, it follows that the quotient $L(1, \pi\otimes\chi_d)/L_S(1, \pi\otimes\chi_d) = g_{S,A}$ is independent of d. (This constant gives the dependence of the distribution function on the ray class A.) Let $L_S(s, \pi \boxplus \tilde{\pi})^k = \sum b_k(m)|m|^{-s}$. By Theorem 1.1 applied to $\pi^{\boxplus k} \boxplus \tilde{\pi}^{\boxplus k}$, for each $k \in \mathbb{N}$ there exists a constant c_k given by the Euler product

$$c_k = \frac{\kappa |g_{S,A}|^{2k}}{\zeta_F(2)\, h_C} \left(\prod_{v\in S} \frac{\zeta_{F,v}(2)}{\zeta_{F,v}(1)} \right) \prod_{(v,S)=1} \left(1 + \frac{q_v}{q_v+1} \sum_{g=1}^{\infty} \frac{b_k(P_v^{2g})}{q_v^{2g}} \right)$$

such that

$$\lim_{X\to\infty} M_k[\mu_X] = c_k.$$

Now

$$b_k(P_v^m) = \sum_{i_1+\cdots+i_k=m} b_1(P_v^{i_1}) \cdots b_1(P_v^{i_k}).$$

If $\pi = \boxplus \pi_j$ and each π_j is tempered, we have

$$|b_1(P_v^i)| \leq p_{2r}(i),$$

where $p_{2r}(i)$ is the partition function in Section 2. Hence

$$|b_k(P_v^m)| \leq \sum_{i_1+\cdots+i_k=m} p_{2r}(i_1)\cdots p_{2r}(i_k) = p_{2rk}(m).$$

The estimate

$$c_k << \exp(crk \log\log(rk))$$

now follows easily, see e.g. Luo [Lu]. By the theorem of Fréchet and Shohat described above, Corollary 1.3 holds. ■

We remark that this last estimate shows that the moment generating function Ψ is entire.

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