

DETERMINATION OF A GL_3 CUSPFORM BY TWISTS OF CENTRAL L -VALUES

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ABSTRACT. Let π be a self-contragredient cuspidal automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$. We show that if the symmetric square L -function of π has a pole at $s = 1$, then π is determined by central values of quadratic twists of its L -function. That is, if π' is another cuspidal automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$ for which $L(\frac{1}{2}, \pi \otimes \chi) = L(\frac{1}{2}, \pi' \otimes \chi)$ for sufficiently many quadratic characters χ , then $\pi \simeq \pi'$.

1. INTRODUCTION

Let π be a self-contragredient cuspidal automorphic representation of $GL_3(\mathbb{A}_{\mathbb{Q}})$ and χ a Dirichlet character. Then the twisted L -function $L(s, \pi \otimes \chi)$ initially defined for $\Re(s) > 1$, is known to have an analytic continuation to \mathbb{C} and to satisfy a certain functional equation relating the values at s to those at $1 - s$. The main result of this paper is that knowledge of these values at the central point $s = 1/2$ for twists by quadratic Dirichlet characters, at least when the symmetric square L -function $L(s, \pi, \text{sym}^2)$ has a pole at $s = 1$, is enough to determine π . Precisely, we prove

Theorem 1.1. *Let π and π' be two self-contragredient cuspidal automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$. Suppose that $L(s, \pi, \text{sym}^2)$ has a pole at $s = 1$. Fix an integer M and let X be the set of all quadratic Dirichlet characters of conductor relatively prime to M . If there exists a nonzero constant κ such that*

$$(1.1) \quad L(\tfrac{1}{2}, \pi \otimes \chi) = \kappa L(\tfrac{1}{2}, \pi' \otimes \chi)$$

for all $\chi \in X$, then $\pi \simeq \pi'$.

Remarks

- (1) The L -functions in (1.1) are the full automorphic L functions, including the archimedean component. However, as will be clear from the proof of the theorem in section 5, if S is any finite set of places of \mathbb{Q} (possibly including the infinite place), the conclusion of the theorem is still true if (1.1) is replaced by

$$L^S(\tfrac{1}{2}, \pi \otimes \chi) = \kappa L^S(\tfrac{1}{2}, \pi' \otimes \chi)$$

for all $\chi \in X$. Here

$$L^S(s, \pi) = \prod_{v \notin S} L(s, \pi_v).$$

Later, we will also use the notation

$$L_S(s, \pi) = \prod_{v \in S} L(s, \pi_v).$$

- (2) The condition $L(s, \pi, \text{sym}^2)$ has a pole at $s = 1$ is satisfied when π is the Gelbart-Jacquet lift [4] of an automorphic representation on $GL_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character. In fact, the converse is true as well. By the work of Ginzburg-Rallis-Soudry (see e.g. [5]), for an irreducible cuspidal automorphic representation π of GL_3 with the partial symmetric square L -function $L^S(s, \pi, \text{sym}^2)$ having a pole at $s = 1$, there exists an irreducible cuspidal automorphic representation σ of $Sp_2 = SL_2$, which lifts functorially to π . We thank Dihua Jiang and David Ginzburg for clarifying this point for us. This fact will make the sieving argument of section 3 somewhat easier.

For holomorphic newforms of congruence subgroups of $SL_2(\mathbb{Z})$, the analogous result was proved by Luo and Ramakrishnan [12]. Their idea is to consider the twisted averages of the form

$$\sum_{d < X} L(\tfrac{1}{2}, f, \chi_d) \chi_d(r)$$

and show that asymptotics for these expressions as $X \rightarrow \infty$ involve the Hecke eigenvalues of f . We prove our result by a similar method. However, the averaging process for a GL_3 cuspform is more delicate, and we use the method of double Dirichlet series, rather than the method of the approximate functional equation used in [12]. The double Dirichlet approach on GL_3 was first carried out by Bump-Friedberg-Hoffstein [2] and Diaconu-Goldfeld-Hoffstein [3]. We rely heavily on the results of these papers.

As the base field is \mathbb{Q} , it is likely that the method of the approximate functional equation will work here as well. Our reason for using the multiple Dirichlet series approach is that it provides a considerably simpler framework and makes the analysis quite easy. The extension to $GL_3(\mathbb{A}_K)$ for K an arbitrary number field, however, remains elusive from the point of view of both approaches.

One goal of this paper is to illustrate the source of this difficulty in our method. The problem arises from the need to apply a “Lindelöf-on-average” bound (Lemma 3.2) to carry out the sieving process of section 3. We establish this bound by appealing to a character sum estimate of Heath-Brown [6], which is valid only over \mathbb{Q} . There are two possible ways to solve this problem: first, establish Lemma 3.2 over an arbitrary base field, or second, prove a “uniqueness” result for the finite Euler products $P_{d_0, d_1}^{(\psi_1)}(1/2)$ of section 2, which would obviate the need to sieve altogether. The second method would be preferable, but the first is of great interest in its own right. A recent illustration of the utility of the method of multiple Dirichlet series in working over number fields is given by J. Li [11] in his thesis, where the original result of Luo and Ramakrishnan is extended to forms on $GL_2(\mathbb{A}_K)$, for K an arbitrary number field. Such a result does not appear to be accessible by the method of the approximate functional equation and large sieve.

In section 2, we review some of the results of Bump-Friedberg-Hoffstein [2] and construct the relevant double Dirichlet series. In section 3 we indicate how to sieve the double Dirichlet series of the previous section in order to obtain a series summed over only square-free discriminants d . In the final two sections we prove the main theorem.

The authors thank S. Friedberg and J. Hoffstein for their encouragement and comments on an earlier version of this paper. We also thank D. Bump, D. Ginzburg, J. Hundley, D. Jiang and Y. Tian for their advice. Finally, we are grateful to the

referee and editor for several comments improving the overall exposition of the paper.

2. ANALYTIC CONTINUATION OF A DOUBLE DIRICHLET SERIES

Let π be a cuspidal automorphic representation of $GL_3(\mathbb{A}_{\mathbb{Q}})$ of conductor N . Let S be a finite set of places of \mathbb{Q} including 2 and the archimedean place, such that π is unramified outside of S . We write S as the disjoint union $S = S_{fin} \cup \{\infty\}$. Let $M = \prod_{p \in S_{fin}} p$. Let $R = (\mathbb{Z}/4M\mathbb{Z})^{\times} \otimes \mathbb{Z}/2\mathbb{Z}$. Characters in the dual group \hat{R} will be used to sieve out congruence classes. We note that characters in \hat{R} are precisely the quadratic characters of $(\mathbb{Z}/4M\mathbb{Z})^{\times}$. For the purposes of constructing L -functions below, we identify a character in \hat{R} with the primitive quadratic Dirichlet character which induces it. Thus, given a divisor l of $4M$, there is exactly one character in \hat{R} of conductor l if 8 does not divide l , and exactly two characters in \hat{R} of conductor l if 8 does divide l .

We recall that L^S and L_S were defined in the remark following the statement of Theorem 1.1. Taking out the archimedean component from the L -function of π , we have the L -series

$$\begin{aligned} L^{\infty}(s, \pi) &= \sum_{m \geq 1} c_m m^{-s} \\ &= \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1} \left(1 - \frac{\gamma_p}{p^s}\right)^{-1}, \end{aligned}$$

the Euler product being taken over all primes p of \mathbb{Q} . For χ a Dirichlet character of conductor D relatively prime to M the twisted L -series

$$\begin{aligned} L^{\infty}(s, \pi \otimes \chi) &= \sum_{m \geq 1} \frac{c_m \chi(m)}{m^s} \\ &= \prod_p \left(1 - \frac{\alpha_p \chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_p \chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\gamma_p \chi(p)}{p^s}\right)^{-1}, \end{aligned}$$

has a functional equation given by

$$L^{\infty}(s, \pi \otimes \chi) = \epsilon_{\pi} \tau(\chi)^3 \chi_{\pi}(D) \chi(N) (D^3 N)^{\frac{1}{2}-s} \frac{L^{\infty}(1-s, \tilde{\pi} \otimes \bar{\chi})}{L^{\infty}(s, \pi \otimes \chi)} L^{\infty}(1-s, \tilde{\pi} \otimes \bar{\chi}).$$

Here χ_{π} is the central character of π , $\tau(\chi)$ is the normalized Gauss sum associated to χ and $\epsilon_{\pi} = \epsilon(1/2, \pi)$ is the central value of the ϵ -factor of π . From now on we will assume that π is self-contragredient with trivial central character and that χ is quadratic. In this case, $\tau(\chi)$ is trivial.

In [2], a double Dirichlet series is constructed out of quadratic twists of the L -function of π . To describe this precisely we need some more notation. Let χ_d denote the quadratic Dirichlet character associated to the extension $\mathbb{Q}(\sqrt{d})$ of \mathbb{Q} . For ψ_1, ψ_2 in \hat{R} , it is shown that there exist finite Euler products $P_{d_0, d_1}^{(\psi_1)}(s)$ such that the double Dirichlet series

$$(2.1) \quad Z_M(s, w, \pi; \psi_2, \psi_1) = \sum_{(d, M)=1} \frac{L^S(s, \pi \otimes \chi_{d_0} \psi_1)}{d^w} \psi_2(d_0) P_{d_0, d_1}^{(\psi_1)}(s)$$

has a meromorphic continuation to \mathbb{C}^2 . The sum is taken over positive integers $d = d_0 d_1^2$, with d_0 squarefree. In fact

Theorem 2.1. *Let*

$$\mathcal{P}(s, w) = w(w-1)(3s+w-5/2)(3s+w-3/2).$$

Then $\mathcal{P}(s, w)Z_M(s, w, \pi; \psi_2, \psi_1)$ has an analytic continuation to an entire function of order 1 on \mathbb{C}^2 .

This series has a polar line at $w = 1$ if and only if $\psi_2 = 1$. In this case, for $\Re(s) > 1/2$, the residue at $w = 1$ is computed in [2]

$$(2.2) \quad \text{Res}_{w=1} Z_M(s, w, \pi; 1, 1) = \prod_{p|M} (1 - 1/p) \cdot L^S(2s, \pi, \text{sym}^2) \zeta^S(6s-1).$$

Moreover, the series satisfies certain functional equations as $s \mapsto 1-s$ and $w \mapsto 1-w$, [2]. These are reviewed in the following section.

For our application we need to consider slightly different sums. For a prime number r relatively prime to M , let $\tilde{\chi}_r$ denote the quadratic character with conductor r defined by $\tilde{\chi}_r(*) = \left(\frac{*}{r}\right)$. Let \mathcal{K} be the set of all positive integers d such that $\psi(d) = 1$ for all $\psi \in \hat{R}$.

For $(d, M) = 1$ we have the orthogonality relation

$$(2.3) \quad \frac{1}{|\hat{R}|} \sum_{\psi \in \hat{R}} \psi(d) = \begin{cases} 1 & \text{if } d \in \mathcal{K} \\ 0 & \text{otherwise.} \end{cases}$$

We let $\delta_{\mathcal{K}}$ be the characteristic function of the subset \mathcal{K} , and define

$$Z(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, \psi_1) := \sum_{d \in \mathcal{K}} \frac{L^\infty(s, \pi \otimes \chi_{d_0} \psi_1)}{d^w} \tilde{\chi}_r(d) P_{d_0, d_1}^{(\psi_1)}(s).$$

Proposition 2.2. *The double Dirichlet series $Z(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, \psi_1)$ has a meromorphic continuation to $\Re(s) > 2/5$ and $w \in \mathbb{C}$. In this region, the product*

$$\mathcal{P}(s, w)Z(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, \psi_1)$$

is analytic.

Proof. We express $Z(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, \psi_1)$ as a linear combination of the functions $Z_M(s, w, \pi; \psi_2, \psi_1)$ defined above. Then the stated meromorphic continuation will follow from the known properties of the $Z_M(s, w, \pi; \psi_2, \psi_1)$. Note that for $d \in \mathcal{K}$ and $p \in S$, we have $\chi_d(p) = 1$. Hence, for $\Re(s), \Re(w)$ sufficiently large,

$$\begin{aligned} Z(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, \psi_1) &= \\ \frac{1}{|\hat{R}|} L_{S_{fin}}(s, \pi \otimes \psi_1) \sum_{\psi_2 \in \hat{R}} \sum_{(d, M)=1} \frac{L^S(s, \pi \otimes \chi_{d_0} \psi_1)}{d^w} \tilde{\chi}_r(d) \psi_2(d) P_{d_0, d_1}^{(\psi_1)}(s), \end{aligned}$$

where we have used the orthogonality relation (2.3). Removing the r^{th} -term from the Euler product of $L^S(s, \pi \otimes \chi_d \psi_1)$ and letting $S_r = S \cup \{r\}$, we write the inner sum over d as

$$\begin{aligned} \sum_{(d, rM)=1} \frac{L^{S_r}(s, \pi \otimes \chi_{d_0} \psi_1)}{d^w} \left(\sum_{k \geq 0} \frac{c_{r^k}(\chi_{d_0} \psi_1)(r^k)}{r^{ks}} \right) \tilde{\chi}_r(d) \psi_2(d) P_{d_0, d_1}^{(\psi_1)}(s) \\ = \psi_1(r) L_{1,r}(s) \cdot Z_{Mr}(s, w, \pi; \psi_2, \psi_1) + L_{2,r}(s) \cdot Z_{Mr}(s, w, \pi; \psi_2 \tilde{\chi}_r, \psi_1), \end{aligned}$$

say, where

$$L_{1,r}(s) = \sum_{k \geq 0} \frac{c_r^{2k+1}}{r^{(2k+1)s}}, \quad L_{2,r}(s) = \sum_{k \geq 0} \frac{c_r^{2k}}{r^{2ks}}.$$

Thus $Z(s, w, \pi; \tilde{\chi}_r \delta_K, \psi_1) =$

$$(2.4) \quad \frac{1}{|\hat{R}|} L_{S_{fin}}(s, \pi \otimes \psi_1) \sum_{\psi_2 \in \hat{R}} (\psi_1(r) L_{1,r}(s) \cdot Z_{Mr}(s, w, \pi; \psi_2, \psi_1) + L_{2,r}(s) Z_{Mr}(s, w, \pi; \psi_2 \tilde{\chi}_r, \psi_1)).$$

To complete the proof of the Proposition, it remains to establish the analytic continuation of $L_{i,r}(s)$, ($i = 1, 2$) to $\Re(s) > 2/5$. This is done in the Lemma below. \square

Lemma 2.3. *The series $L_{i,r}(s)$, ($i = 1, 2$) converge absolutely for $\Re(s) > 2/5$. Moreover, we have the explicit representations*

$$L_{1,r}(s) = \frac{c_r + r^{-2s}}{r^s(1 - \alpha_r^2 r^{-2s})(1 - \beta_r^2 r^{-2s})(1 - \gamma_r^2 r^{-2s})}$$

and

$$L_{2,r}(s) = \frac{1 + c_r r^{-2s}}{(1 - \alpha_r^2 r^{-2s})(1 - \beta_r^2 r^{-2s})(1 - \gamma_r^2 r^{-2s})}.$$

Proof. The absolute convergence of both series follows from the bound of Luo, Rudnick and Sarnak [13]:

$$c_n \ll_\epsilon n^{\frac{2}{5} + \epsilon},$$

for any $\epsilon > 0$. To evaluate the sum $L_{1,r}(s)$, we begin by writing the Fourier coefficient $c_{r,k}$ in terms of the Satake parameters $\alpha = \alpha_r, \beta = \beta_r, \gamma = \gamma_r$:

$$c_{r,k} = \frac{\begin{vmatrix} \alpha^{k+2} & \beta^{k+2} & \gamma^{k+2} \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{vmatrix}}.$$

We expand the determinants and evaluate the sum $L_{i,r}(s)$ as a linear combination of geometric series. After some algebraic simplifications using the relations

$$c_r = \alpha + \beta + \gamma = \alpha\beta + \alpha\gamma + \beta\gamma$$

we arrive at the desired result. \square

Remark The products in the denominator can be similarly evaluated to give rational expressions of c_r :

$$(2.5) \quad \begin{aligned} (1 - \alpha_r^2 r^{-2s})(1 - \beta_r^2 r^{-2s})(1 - \gamma_r^2 r^{-2s}) &= \\ 1 - \frac{c_r^2 - 2c_r}{r^{2s}} + \frac{c_r^2 - 2c_r}{r^{4s}} - \frac{1}{r^{6s}}. \end{aligned}$$

3. SIEVING THE DOUBLE DIRICHLET SERIES

In this section we show that the “imperfect” double Dirichlet series without weighting polynomials $P_{d_0, d_1}^{(\psi_1)}(s)$ has a meromorphic continuation to suitably large domain.

Proposition 3.1. *The series*

$$Z^b(s, w, \pi; \tilde{\chi}_r \delta_K, 1) = \sum_{\substack{d_0 \in K, d_0 \text{ sq. free} \\ d_0 > 0}} \frac{L^\infty(s, \pi \otimes \chi_{d_0})}{d_0^w} \tilde{\chi}_r(d_0)$$

has a meromorphic continuation to a tube domain in \mathbb{C}^2 containing the point $(s, w) = (1/2, 1)$. More precisely, the product

$$\mathcal{P}(s, w) Z(s, w, \pi; \tilde{\chi}_r \delta_K, 1)$$

is analytic in the union of the two regions

$$\{\Re(s) > \tfrac{1}{2}, \Re(w) > \tfrac{119}{124}\} \text{ and } \{0 \leq \Re(s) \leq \tfrac{1}{2}, \Re(w) > -\tfrac{191}{62}\Re(s) + \tfrac{5}{2}\}.$$

The proposition is proved by a sieving argument similar to that used in [3]. We assume familiarity with the methods of [3] and merely indicate in this section where modifications are needed. What makes the argument more difficult in our situation is the lack of a sufficiently powerful “Lindelöf-on-average” result for the mean square of twisted central values of the L -function of a GL_3 cuspform. The best that one can presently do is

Lemma 3.2. *Let π be a self-adjoint cuspidal automorphic representation of $GL_r(\mathbb{A}_{\mathbb{Q}})$. For all $\epsilon > 0$, we have the estimate*

$$\sum_{|d| < x} |L(\tfrac{1}{2}, \pi \otimes \chi_d)|^2 \ll_{\epsilon} x^{\frac{r}{2} + \epsilon}.$$

This Lemma follows readily from the following character sum estimate of Heath-Brown’s [6].

Lemma 3.3. *Let N, Q be positive integers, and let a_1, a_2, \dots, a_N be arbitrary complex numbers. Then, for any $\epsilon > 0$,*

$$\sum_{|d| \leq Q} \left| \sum_{n \leq N} a_n \chi_d(n) \right|^2 \ll_{\epsilon} (QN)^{\epsilon} (Q + N) \sum_{\substack{n_1, n_2 \leq N \\ n_1 \cdot n_2 \equiv \square}} |a_{n_1} a_{n_2}|.$$

It is for this reason that we are forced to assume that the base field is \mathbb{Q} . It would be of interest to establish an analogue of the Lemma over an arbitrary base field. With Lemma 3.2 in hand, what is needed for the sieving is the modified version of Proposition 4.12 of [3] given below.

Proposition 3.4. *Let $w = \nu + it$. Let $\epsilon > 0, -\epsilon - \frac{1}{4} \leq \nu$. Let $\psi_1, \psi_2 \in \hat{R}$. We will denote the conductor of $\psi_i \in \hat{R}$ by l_i . The function $Z_M(1/2, w, \pi; \psi_2, \psi_1)$ is an analytic function of w , except for possible poles at $w = \frac{3}{4}$ and $w = 1$. If $(l_1, l_2) = 1, 2, 4$ or 8 and $|t| > 1$, then it satisfies the upper bounds*

$$Z_M\left(\frac{1}{2}, \nu + it, \pi; \psi_2, \psi_1\right) \ll_{\epsilon} M^{\epsilon},$$

for $\frac{5}{4} + \epsilon < \nu$, and

$$(3.1) \quad Z_M \left(\frac{1}{2}, -\frac{1}{4} - \epsilon + it, \pi; \psi_2, \psi_1 \right) \ll_{\epsilon, \pi, t} M^{5+v(\epsilon)} \sum_{\psi_3 \in \hat{R}} \sum_{\substack{d_0 \text{ sq. free} \\ (d_0, M)=1}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_3)|}{d_0^{\frac{5}{4}+\epsilon} l_3^{\frac{1}{4}}}.$$

The function $v(\epsilon)$ is computable and satisfies $v(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. The fact that this function is analytic in w , except for possible poles at $w = \frac{3}{4}$ and $w = 1$, is proved in [2] and [3].

To prove the first estimate, fix $x > 1$. Then,

$$\begin{aligned} & \sum_{\substack{d < x \\ (d, M)=1}} \frac{L^S(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_1)}{d^{\nu+it}} \psi_2(d_0) P_{d_0, d_1}^{(\psi_1)}(\frac{1}{2}) \\ & \ll_{\epsilon} M^{\epsilon} \sum_{\substack{d_0 < x, d_0 \text{ sq. free} \\ (d_0, M)=1}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_1)|}{d_0^{\nu}} \sum_{\substack{d_1^2 < \frac{x}{d_0} \\ (d_1, M)=1}} \frac{|P_{d_0, d_1}^{(\psi_1)}(\frac{1}{2})|}{d_1^{2\nu}} \\ & \ll_{\epsilon} M^{\epsilon} \sum_{\substack{d_0 < x, d_0 \text{ sq. free} \\ (d_0, M)=1}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_1)|}{d_0^{\nu}} \sum_{\substack{d_1=1 \\ (d_1, M)=1}}^{\infty} \frac{|P_{d_0, d_1}^{(\psi_1)}(\frac{1}{2})|}{d_1^{2\nu}}, \end{aligned}$$

where the last inner sum is absolutely convergent for $\nu > \frac{5}{4}$, and it is $\ll 1$ independent of d_0 and M . It follows that

$$(3.2) \quad Z_M(\frac{1}{2}, \nu + it, \pi; \psi_2, \psi_1) \ll_{\epsilon, \nu} M^{\epsilon} \sum_{\substack{d_0 \text{ sq. free} \\ (d_0, M)=1}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_1)|}{d_0^{\nu}} \quad (\nu > \frac{5}{4}).$$

The absolute convergence of the series in the right hand side, for $\nu > 5/4$, can be easily justified by applying the Cauchy-Schwarz inequality and the estimate in Lemma 3.2.

To justify (3.1) we define two involutions on \mathbb{C}^2 :

$$\alpha : (s, w) \rightarrow (1 - s, 3s + w - \frac{3}{2}) \quad \text{and} \quad \beta : (s, w) \rightarrow (s + w - \frac{1}{2}, 1 - w).$$

If $\vec{Z}_M(s, w, \pi)$ denotes the column vector whose entries are $Z_M(s, w, \pi; \psi_2, \psi_1)$ with $\psi_1, \psi_2 \in \hat{R}$, then by Propositions 4.2 and 4.3 in [3], there exist matrices $\Psi_M(s)$ and $\Phi_M(w)$ such that

$$\vec{Z}_M(s, w, \pi) = \Psi_M(s) \cdot \vec{Z}_M(\alpha(s, w), \pi) \quad \vec{Z}_M(s, w, \pi) = \Phi_M(w) \cdot \vec{Z}_M(\beta(s, w), \pi).$$

Applying the transformation $\beta\alpha\beta\alpha\beta$, one obtains the functional equation:

$$\vec{Z}_M(s, w, \pi) = \mathcal{M}(s, w) \cdot \vec{Z}_M(s, \frac{5}{2} - 3s - w, \pi),$$

where

$$\mathcal{M}(s, w) := \Phi_M(w) \Psi_M(s + w - \frac{1}{2}) \Phi_M(3s + 2w - 2) \Psi_M(2s + w - 1) \Phi_M(3s + w - \frac{3}{2}).$$

We shall need to estimate the entries of the matrix $\mathcal{M}(\frac{1}{2}, -\frac{1}{4} + it)$. To do this, we recall the explicit description of, for instance, the β -functional equation (see (4.18))

in [3]). We continue to let l_i denote the conductor of $\psi_i \in \hat{R}$. If the conductor l_2 of ψ_2 is odd, we have

$$(3.3) \quad \prod_{p|(M/l_2)} (1 - p^{-2+2w}) \cdot Z_M(s, w, \pi; \psi_2, \psi_1) \\ = \frac{1}{2} l_2^{\frac{1}{2}-w} \cdot \sum_{l_3, l_4|(M/l_2)} \mu(l_3) \psi_2(l_3 l_4) l_3^{-w} l_4^{-1+w} \cdot \sum_{a=\pm 1} \frac{L_\infty(1-w, \chi_a \psi_2)}{L_\infty(w, \chi_a \psi_2)} \\ \cdot (Z_M(\beta(s, w), \pi; \psi_2, \psi_1 \psi_3 \psi_4) + a Z_M(\beta(s, w), \pi; \psi_2, \chi_{-1} \psi_1 \psi_3 \psi_4)).$$

Here $\chi_1 \equiv 1$, and χ_{-1} is the character defined by

$$\chi_{-1}(m) = \left(\frac{-4}{m} \right) = \begin{cases} (-1)^{\frac{|m|-1}{2} + \frac{\text{sgn}(m)-1}{2}} & \text{if } m \equiv 1 \pmod{2} \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

When l_2 is even, we have a similar expression. In fact, just the behavior at the finite place 2 changes.

Using Stirling's formula, we obtain the estimate

$$(3.4) \quad \prod_{p|(M/l_2)} (1 - p^{-2+2\Re(w)}) \cdot Z_M(s, w, \pi; \psi_2, \psi_1) \\ \ll_w l_2^{\frac{1}{2}-\Re(w)} \cdot \sum_{l_3, l_4|(M/l_2)} l_3^{-\Re(w)} l_4^{-1+\Re(w)} \cdot |Z_M(\beta(s, w), \pi; \psi_2, \psi_1 \psi_3 \psi_4)|.$$

A similar estimate corresponding to the α -functional equation can also be established:

$$(3.5) \quad Z_M(s, w, \pi; \psi_2, \psi_1) \cdot \prod_{p|(M/l_1)} (1 - |\alpha_p|^2 p^{-2+2s}) (1 - |\beta_p|^2 p^{-2+2s}) (1 - |\gamma_p|^2 p^{-2+2s}) \\ \ll_s l_1^{\frac{3}{2}-3\Re(s)} \cdot \sum_{l_\alpha|(M/l_1)} |\alpha_{l_\alpha}| l_\alpha^{-\Re(s)} \cdot \sum_{l_\beta|(M/l_1)} |\beta_{l_\beta}| l_\beta^{-\Re(s)} \\ \cdot \sum_{l_\gamma|(M/l_1)} |\gamma_{l_\gamma}| l_\gamma^{-\Re(s)} \cdot \sum_{l_{\tilde{\alpha}}|(M/l_1)} |\alpha_{l_{\tilde{\alpha}}}| l_{\tilde{\alpha}}^{-1+\Re(s)} \\ \cdot \sum_{l_{\tilde{\beta}}|(M/l_1)} |\beta_{l_{\tilde{\beta}}}| l_{\tilde{\beta}}^{-1+\Re(s)} \cdot \sum_{l_{\tilde{\gamma}}|(M/l_1)} |\gamma_{l_{\tilde{\gamma}}}| l_{\tilde{\gamma}}^{-1+\Re(s)} \\ \cdot |Z_M(\alpha(s, w), \pi; \psi_2 \psi_\alpha \psi_\beta \psi_\gamma \psi_{\tilde{\alpha}} \psi_{\tilde{\beta}} \psi_{\tilde{\gamma}}, \psi_1)|$$

where we have set

$$(3.6) \quad \alpha_{l_\alpha} = \prod_{p|l_\alpha} \alpha_p, \quad \beta_{l_\beta} = \prod_{p|l_\beta} \beta_p, \quad \gamma_{l_\gamma} = \prod_{p|l_\gamma} \gamma_p,$$

and similarly for $\alpha_{l_{\tilde{\alpha}}}$, $\beta_{l_{\tilde{\beta}}}$, $\gamma_{l_{\tilde{\gamma}}}$.

The functional equations are more cleanly expressed in matrix notation. We will see that the corresponding matrices representing the right hand sides of the estimates (3.4) and (3.5) decompose as tensor products over the primes dividing M . To this end, write $4M = M_0 M_1$, where M_0 is a power of 2 and M_1 is odd and square-free. For the proof of Proposition 3.4, we need to bound Z_M in terms of M . Because of the tensor product structure we will exhibit, we will see that the bounds

we obtain are multiplicative in M . For this reason, we may ignore a finite number of primes (e.g. the prime 2) and we may assume M_1 is prime.

Under these simplifications, if $M_1 = p$ is prime, we fix

$$\vec{Z}_p(s, w, \pi) = \begin{pmatrix} Z_p(s, w, \pi; 1, 1) \\ Z_p(s, w, \pi; 1, \psi) \\ Z_p(s, w, \pi; \psi, 1) \\ Z_p(s, w, \pi; \psi, \psi) \end{pmatrix},$$

where $\psi = \chi_p$ is the character associated to the quadratic extension $\mathbb{Q}(\sqrt{p'})$ of \mathbb{Q} , where $p' = (-1)^{\frac{p-1}{2}}p$. We now define two matrices which will be used to build up the matrices which represent the right hand sides of the estimates (3.4) and (3.5) for the β and α -functional equations, respectively. Let

$$\Phi'_p(w) = \begin{pmatrix} \frac{1+p^{-1}}{1-p^{-2+2w}} & \frac{p^{-w}+p^{-1+w}}{1-p^{-2+2w}} & 0 & 0 \\ \frac{p^{-w}+p^{-1+w}}{1-p^{-2+2w}} & \frac{1+p^{-1}}{1-p^{-2+2w}} & 0 & 0 \\ 0 & 0 & p^{\frac{1}{2}-w} & 0 \\ 0 & 0 & 0 & p^{\frac{1}{2}-w} \end{pmatrix}$$

$$\Psi'_p(s) = \begin{pmatrix} u(s) & 0 & v(s) & 0 \\ 0 & p^{\frac{3}{2}-3s} & 0 & 0 \\ v(s) & 0 & u(s) & 0 \\ 0 & 0 & 0 & p^{\frac{3}{2}-3s} \end{pmatrix},$$

where $u(s)$ and $v(s)$ are given by

$$u(s) = \frac{*}{(1 - |\alpha_p|^2 p^{-2+2s})(1 - |\beta_p|^2 p^{-2+2s})(1 - |\gamma_p|^2 p^{-2+2s})},$$

$$v(s) = \frac{**}{(1 - |\alpha_p|^2 p^{-2+2s})(1 - |\beta_p|^2 p^{-2+2s})(1 - |\gamma_p|^2 p^{-2+2s})}$$

with

$$\begin{aligned} * &= [1 + (|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})p^{-2+2s} + (|\alpha_p| + |\beta_p| + |\gamma_p|)^2 p^{-1} \\ &+ (|\alpha_p| + |\beta_p| + |\gamma_p|)p^{-3+2s} + (|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})p^{-2s} \\ &+ (|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})^2 p^{-2} + (|\alpha_p| + |\beta_p| + |\gamma_p|)p^{-1-2s} + p^{-3}] \end{aligned}$$

and

$$\begin{aligned} ** &= [(|\alpha_p| + |\beta_p| + |\gamma_p|)p^{-1+s} + p^{-3+3s} + (|\alpha_p| + |\beta_p| + |\gamma_p|)p^{-s} \\ &+ (|\alpha_p| + |\beta_p| + |\gamma_p|)(|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})p^{-2+s} \\ &+ (|\alpha_p| + |\beta_p| + |\gamma_p|)(|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})p^{-1-s} \\ &+ (|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})p^{-3+s} + (|\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1})p^{-2-s} + p^{-3s}]. \end{aligned}$$

(We digress a moment to explain the apparent asymmetry in considering only the quadratic extensions $\mathbb{Q}(\sqrt{p'})$. The characters corresponding to the extensions $\mathbb{Q}(\sqrt{-p'})$ will appear after we tensor with the matrices corresponding to the prime 2. These 16×16 matrices act on the 16-dimensional vector whose components are $Z_2(s, w, \pi; \psi_1, \psi_2)$, with ψ_1 and ψ_2 each being one of the 4 primitive quadratic characters of conductor a power of 2. Fortunately, we will not need to write down these matrices as we are only interested in bounds for Z_M in terms of M , as M grows large. Therefore, as remarked above, we can ignore finitely many small primes.)

For M_1 odd and square-free, one can compute upper bounds for the matrices $\Psi_{M_1}(s)$ and $\Phi_{M_1}(w)$ as follows. Let p be a prime divisor of M_1 , and let V_p be a complex vector space spanned by a basis

$$\{e_p(l_2, l_1) : l_1, l_2 | p\}.$$

We consider two additional complex vector spaces V_p^α and V_p^β spanned by

$$\{e_p^\alpha(l_2, l_1) : l_1, l_2 | p\} \quad \{e_p^\beta(l_2, l_1) : l_1, l_2 | p\},$$

respectively, and let

$$\Psi'_p : V_p \longrightarrow V_p^\alpha \quad \Phi'_p : V_p \longrightarrow V_p^\beta$$

be the linear maps corresponding to $\Psi'_p(s)$ and $\Phi'_p(w)$. For instance, Φ'_p is described by

$$e_p(l_2, l_1) \mapsto \prod_{\substack{q|(p/l_2) \\ q \neq 1}} (1 - q^{-2+2w})^{-1} \cdot l_2^{\frac{1}{2}-w} \cdot \sum_{l_3, l_4 | (p/l_2)} l_3^{-w} l_4^{-1+w} \cdot e_p^\beta(l_2, l_1 l_3 l_4).$$

Here, we made the convention that $e_p^\beta(a, n^2 b) = e_p^\beta(a, b)$.

Let

$$V_{M_1} := \bigotimes_{p|M_1} V_p; \quad V_{M_1}^\alpha := \bigotimes_{p|M_1} V_p^\alpha; \quad V_{M_1}^\beta := \bigotimes_{p|M_1} V_p^\beta.$$

Then, $\Psi'_{M_1} = \bigotimes_{p|M_1} \Psi'_p$ and $\Phi'_{M_1} = \bigotimes_{p|M_1} \Phi'_p$.

Now, for $M_1 = p$, we have

$$\Phi_{M_1}(-\tfrac{1}{4}) \ll \Phi'_{M_1}(-\tfrac{1}{4}) \sim \begin{pmatrix} 1 & p^{\frac{1}{4}} & 0 & 0 \\ p^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & 0 & p^{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & p^{\frac{3}{4}} \end{pmatrix},$$

$$\Psi_{M_1}(-\tfrac{1}{4}) \ll \Psi'_{M_1}(-\tfrac{1}{4}) \sim \begin{pmatrix} c'_p p^{\frac{1}{2}} & 0 & p^{\frac{3}{4}} & 0 \\ 0 & p^{\frac{9}{4}} & 0 & 0 \\ p^{\frac{3}{4}} & 0 & c'_p p^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & p^{\frac{9}{4}} \end{pmatrix},$$

and

$$\Phi_{M_1}(-1) \ll \Phi'_{M_1}(-1) \sim \begin{pmatrix} 1 & p & 0 & 0 \\ p & 1 & 0 & 0 \\ 0 & 0 & p^{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & p^{\frac{3}{2}} \end{pmatrix} \quad \text{as } p \rightarrow \infty,$$

where

$$c'_p := |\alpha_p|^{-1} + |\beta_p|^{-1} + |\gamma_p|^{-1}.$$

Recall that by the remark (2) made after the statement of Theorem 1.1, the cuspidal automorphic representation π is a Gelbart-Jacquet lift. It follows that

$$c'_p \ll p^{\frac{2}{9}}$$

by the bound of Kim and Shahidi in [9]. In fact, the weaker exponent $5/28$ obtained by Bump, Duke, Hoffstein and Iwaniec in [1] would suffice for our purposes. It

follows easily that

$$\mathcal{M}(\tfrac{1}{2}, -\tfrac{1}{4}) \ll \begin{pmatrix} p^5 & p^{\frac{19}{4}} & p^5 & 0 \\ p^{\frac{19}{4}} & p^{\frac{9}{2}} & p^{\frac{19}{4}} & 0 \\ p^5 & p^{\frac{19}{4}} & p^{\frac{40}{9}} & 0 \\ 0 & 0 & 0 & p^{\frac{15}{2}} \end{pmatrix}.$$

To prove (3.1) consider the first three entries of $\vec{Z}_p(s, w, \pi)$. Recall that in the statement of the proposition, we have the condition $(l_1, l_2) = 1, 2, 4$ or 8 . For example, we have that

$$\begin{aligned} |Z_p(\tfrac{1}{2}, -\tfrac{1}{4} - \epsilon + it, \pi; 1, 1)| &\ll p^{v(\epsilon)} \cdot (p^5 |Z_p(\tfrac{1}{2}, \tfrac{5}{4} + \epsilon - it, \pi; 1, 1)| \\ &+ p^{\frac{19}{4}} |Z_p(\tfrac{1}{2}, \tfrac{5}{4} + \epsilon - it, \pi; 1, \psi)| + p^5 |Z_p(\tfrac{1}{2}, \tfrac{5}{4} + \epsilon - it, \pi; \psi, 1)|). \end{aligned}$$

The factor $v(\epsilon)$ is a linear function of ϵ coming from repeated applications of the functional equations. For example, when $w = -1/4 - \epsilon$ and $M_1 = p$ is prime, the estimate (3.4) would produce a p^ϵ .

For general M , we can use the tensor product decomposition of Ψ_M and Φ_M obtaining

$$\begin{aligned} |Z_M(\tfrac{1}{2}, -\tfrac{1}{4} - \epsilon + it, \pi; \psi_2, \psi_1)| \\ \ll M^{v(\epsilon)} \sum_{\substack{\psi_3, \psi_4 \in \hat{R} \\ (l_3, l_4)=1}} M^5 l_3^{-\frac{1}{4}} |Z_M(\tfrac{1}{2}, \tfrac{5}{4} + \epsilon - it, \pi; \psi_4, \psi_3)|. \end{aligned}$$

Now, the estimate (3.1) follows immediately from (3.2). \square

To conclude the section, we now prove Proposition 3.1. We have

$$Z^b(s, w, \pi; \tilde{\chi}_r \delta_K, 1) = \sum_{\substack{d_0 \in \mathcal{K}, d_0 \text{ sq. free} \\ d_0 > 0}} \frac{L^\infty(s, \pi \otimes \chi_{d_0})}{d_0^w} \tilde{\chi}_r(d_0).$$

As in the proof of Proposition 2.2, we can express $Z^b(s, w, \pi; \tilde{\chi}_r \delta_K, 1)$ as a finite linear combination of the series

$$Z_{Mr}^b(s, w, \pi; \psi_2, 1) = \sum_{\substack{(d_0, Mr)=1 \\ d_0 \text{ sq. free}}} \frac{L^S(s, \pi \otimes \chi_{d_0})}{d^w} \psi_2(d_0)$$

Since r will be fixed for the rest of the section, we relabel Mr as M' . We now indicate how to analytically continue $Z_{M'}^b(s, w, \pi; \psi_2, 1)$ to a region containing the point $(1/2, 1)$.

First, we write

$$Z_{M'}^b(s, w, \pi; \psi_2, 1) = \sum_{(q, M')=1} \mu(q) Z_{M'}(s, w, \pi; \psi_2, 1; q),$$

where the sum is over square-free q and

$$Z_{M'}(s, w, \pi; \psi_2, 1; q) := \sum_{\substack{(d_0 d_1, M')=1 \\ d_1 \equiv 0 \pmod{q}}} \frac{L^S(s, \pi \otimes \chi_{d_0})}{d^w} \psi_2(d_0) P_{d_0, d_1}(s).$$

In [2] an explicit description of the weighting polynomials $P_{d_0, d_1}(s)$ is given. We need an estimate in the q -aspect for $Z_{M'}(\frac{1}{2}, w, \pi; \psi_2, 1; q)$ in a strip $-\frac{1}{4} - \epsilon < \Re(w) < \frac{5}{4} + \epsilon$ with a small $\epsilon > 0$. From the computations of [2], we know that

$$P_{d_0, d_1}(s) = \prod_{p^l \parallel d_1} P_{d_0, p^l}(s)$$

and, for p prime, we have the following bound independent of d_0

$$\sum_{l=0}^{\infty} \frac{P_{d_0, p^l}(s)}{p^{ls}} = 1 + p^{-2\Re(w)} O\left(\frac{|c_p^{(2)}| + |c_p|}{p^{2\Re(s)-1}} + 1\right)$$

where $c_p^{(2)}$ are the coefficients of $L(s, \pi, \text{sym}^2)$. Therefore for $\Re(w) > 5/4$ and $\Re(s) \geq 1/2$, we have

$$Z_{M'}(s, w, \pi; \psi_2, 1; q) \ll q^{-2\Re(w)} \cdot (|c_q^{(2)}| + |c_q|) q^{1-2\Re(s)} + 1.$$

To obtain an estimate for $\Re(w) < -1/4$ and $\Re(s) \geq 1/2$, we write

$$(3.7) \quad Z_{M'}(s, w, \pi; \psi_2, 1; q) = \sum_{l|q} \mu(l) Z_{M'}^{(l)}(s, w, \pi; \psi_2, 1),$$

where

$$Z_{M'}^{(l)}(s, w, \pi; \psi_2, 1) := \sum_{\substack{(d_0, M')=1 \\ (d_1, M'l)=1}} \frac{L^S(s, \pi \otimes \chi_{d_0})}{d^w} \psi_2(d_0) P_{d_0, d_1}(s).$$

The point is that one can decompose $Z_{M'}^{(l)}$ as a linear combination of the functions $Z_{M'}(s, w, \pi; \psi_2, \psi_1)$.

Proposition 3.5. *We have*

$$(3.8) \quad Z_{M'}^{(l)}(s, w, \pi; \psi_2, 1) \cdot \prod_{p|l} \left(1 - \frac{\alpha_p^2}{p^{2s}}\right) \left(1 - \frac{\beta_p^2}{p^{2s}}\right) \left(1 - \frac{\gamma_p^2}{p^{2s}}\right) = \\ \frac{1}{2} \sum_{l_3|l} \frac{1}{l_3^w} \prod_{p|l_3} \left(1 - \frac{\alpha_p^2}{p^{2s}}\right) \left(1 - \frac{\beta_p^2}{p^{2s}}\right) \left(1 - \frac{\gamma_p^2}{p^{2s}}\right) \cdot \sum_{m_1, m_2, m_3 | (l/l_3)} \frac{\chi_{l_3}(m_1 m_2 m_3) \alpha_{m_1} \beta_{m_2} \gamma_{m_3}}{(m_1 m_2 m_3)^s} \\ \times [Z_{M'l}(s, w, \pi; \psi_2 \chi_{m_1 m_2 m_3}, \chi_{l_3}) + Z_{M'l}(s, w, \pi; \psi_2 \chi_{-m_1 m_2 m_3}, \chi_{l_3}) \\ + \chi_{-1}(m_1 m_2 m_3) Z_{M'l}(s, w, \pi; \psi_2 \chi_{m_1 m_2 m_3}, \chi_{l_3}) \\ - \chi_{-1}(m_1 m_2 m_3) Z_{M'l}(s, w, \pi; \psi_2 \chi_{-m_1 m_2 m_3}, \chi_{l_3})].$$

We recall that α_m, β_m and γ_m were defined in (3.6). The proof is similar to that of Proposition 4.14 of [3] and will be omitted.

Applying Proposition 3.4, it follows that, for $\Re(w) < -1/4$,

$$Z_{M'}^{(l)}(\frac{1}{2}, w, \pi; \psi_2, 1) \ll (M'q)^{\frac{21}{4}+\epsilon} \sum_{\psi_3 \in \hat{R}_q} \sum_{d_0 \text{ sq. free}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_3)|}{d_0^{1-\Re(w)} l_3^{\frac{1}{4}}},$$

where \hat{R}_q is the dual of $R_q = (\mathbb{Z}/4M'q\mathbb{Z})^\times \otimes \mathbb{Z}/2\mathbb{Z}$ and l_3 is the conductor of ψ_3 . Clearly, the same estimate holds for $Z_{M'}(\frac{1}{2}, w, \pi; \psi_2, 1; q)$.

We now use the Phragmen-Lindelöf principle on the holomorphic function

$$\mathcal{P}(s, w) Z_{M'}(\frac{1}{2}, w, \pi; \psi_2, 1; q).$$

Since Proposition 3.5 and (3.7) allow us to express $Z_{M'}(\frac{1}{2}, w, \pi; \psi_2, 1; q)$ as a finite linear combination of the $Z_{M'}$'s, the fact that the above product is holomorphic and of order 1 follows from Theorem 2.1. Therefore we may apply Phragmen-Lindelöf and obtain the estimate

$$(3.9) \quad Z_{M'}(\tfrac{1}{2}, w, \pi; \psi_2, 1; q) \ll (|c_q^{(2)}| + |c_q| + 1) \cdot q^{\frac{95}{24} - \frac{31}{6}\Re(w) + v(\epsilon)} \cdot \sum_{\psi_3 \in \hat{R}_q} \sum_{d_0 \text{ sq. free}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_3)|}{d_0^{\frac{5}{4} + \epsilon} l_3^{\frac{1}{4}}},$$

for $-\frac{1}{4} - \epsilon < \Re(w) < \frac{5}{4} + \epsilon$. Here $v(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Using the above estimate, it follows that

$$Z_{M'}^\flat(\tfrac{1}{2}, w, \pi; \psi_2, 1) = \sum_{(q, M')=1} \mu(q) Z_{M'}(\tfrac{1}{2}, w, \pi; \psi_2, 1; q)$$

is absolutely bounded by the sum of the three series

$$\Sigma_1 + \Sigma_2 + \Sigma_3,$$

where, for instance,

$$\Sigma_1 = \sum_{\substack{q \text{ sq. free} \\ (q, M')=1}} |c_q^{(2)}| q^{\frac{95}{24} - \frac{31}{6}\Re(w) + v(\epsilon)} \cdot \sum_{\psi_3 \in \hat{R}_q} \sum_{d_0 \text{ sq. free}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \psi_3)|}{d_0^{\frac{5}{4} + \epsilon} l_3^{\frac{1}{4}}}.$$

Each of these sums may be bounded in the same way. As Σ_1 is the most difficult to bound, we provide details in this case only.

We decompose $l_3 = l_1 l_2$ with $l_1 | 4M'$ and $l_2 | q$. If we write $q = l_2 n$, then

$$\Sigma_1 \ll \sum_{l_1 | 4M'} \sum_{\substack{l_2 \text{ sq. free} \\ (l_2, M')=1}} \sum_{d_0 \text{ sq. free}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \chi_{l_1} \chi_{l_2})| \cdot |c_{l_2}^{(2)}|}{d_0^{\frac{5}{4} + \epsilon} l_1^{\frac{1}{4}} l_2^{\frac{31}{6}\Re(w) - \frac{89}{24} - v(\epsilon)}} \sum_{n \geq 1} \frac{|c_n^{(2)}|}{n^{\frac{31}{6}\Re(w) - \frac{95}{24} - v(\epsilon)}}.$$

Fix any $\nu_0 > \frac{119}{124}$. Note that

$$\frac{31}{6}\nu_0 - \frac{95}{24} > 1.$$

Since by the work of Kim [8] and Kim-Shahidi [10], $\text{sym}^2(\pi)$ is automorphic (recall that π is a Gelbart-Jacquet lift), it follows that, for $\Re(w) > \nu_0$ and ϵ sufficiently small, the innermost sum is absolutely convergent. Therefore,

$$\Sigma_1 \ll \sum_{l_1 | 4M'} \sum_{\substack{l_2 \text{ sq. free} \\ (l_2, M')=1}} \sum_{d_0 \text{ sq. free}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_0} \chi_{l_1} \chi_{l_2})| \cdot |c_{l_2}^{(2)}|}{d_0^{\frac{5}{4} + \epsilon} l_1^{\frac{1}{4}} l_2^{\frac{31}{6}\Re(w) - \frac{89}{24} - v(\epsilon)}}.$$

Now, write $d_0 = dd_1$ and $l_2 = dl_0$ with $(d_1, l_0) = 1$. Introducing the Euler factors corresponding to the primes dividing d , and then summing over d , we are reduced to estimating

$$\sum_{l_1 | 4M'} \sum_{\substack{(l_0, d_1)=1 \\ (l_0, M')=1}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_1} \chi_{l_1} \chi_{l_0})| \cdot |c_{l_0}^{(2)}|}{d_1^{\frac{5}{4} + \epsilon} l_1^{\frac{1}{4}} l_0^{\frac{31}{6}\Re(w) - \frac{89}{24} - v(\epsilon)}}.$$

Let $\nu_w := \min\{\frac{5}{4} + \epsilon, \frac{31}{6}\Re(w) - \frac{89}{24} - v(\epsilon)\}$ ($\Re(w) > \nu_0$). Note that $\nu_w > \frac{5}{4}$ for ϵ sufficiently small. Then, for fixed l_1 , we have

$$\sum_{\substack{(l_0, d_1)=1 \\ (l_0, M')=1}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_{d_1} \chi_{l_1} \chi_{l_0})| \cdot |c_{l_0}^{(2)}|}{(d_1 l_0)^{\nu_w}} \ll \sum_{m \text{ sq. free}} \frac{a_m |L(\frac{1}{2}, \pi \otimes \chi_m \chi_{l_1})|}{m^{\nu_w}},$$

where $a_m = \sum_{l_0|m} |c_{l_0}^{(2)}|$. To see that this final series is absolutely convergent, it suffices by the Cauchy-Schwartz inequality to establish that the two series

$$\sum_{m \text{ sq. free}} \frac{a_m^2}{m^{1+\delta}}$$

and

$$\sum_{m \text{ sq. free}} \frac{|L(\frac{1}{2}, \pi \otimes \chi_m \chi_{l_1})|^2}{m^{3/2+\delta}}$$

are absolutely convergent for any $\delta > 0$. The convergence of the first follows from properties of the Rankin-Selberg L -function of $\text{sym}^2(\pi)$ with itself, and the convergence of the second from Lemma 3.2.

This establishes the analytic continuation of $Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ to

$$(3.10) \quad \{\Re(s) \geq \frac{1}{2}, \Re(w) > \frac{119}{124}\}.$$

To complete the proof of Proposition 3.1, we note that, because of the functional equation, we have the bound

$$L(s, \pi \otimes \chi_{d_0}) \ll_{\epsilon} |d_0|^{3/2+\epsilon}$$

for $\Re(s) = 0$. Therefore, $Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ is holomorphic in the tube domain

$$(3.11) \quad \{\Re(s) > 0, \Re(w) > \frac{5}{2}\}.$$

Applying Hartogs' theorem [7] and taking the convex closure of the two tube domains (3.10) and (3.11) completes the proof of Proposition 3.1.

4. COMPUTING THE RESIDUE

The series $Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ has polar lines at $w = 1$ and at $w + 3s = 5/2$ which it inherits from $Z(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$. Except for these two polar lines, $Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ is holomorphic in a neighborhood of the point $(s, w) = (1/2, 1)$. The residue of $Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ at $w = 1$ may be computed from the known residue (2.2) of the $Z_M(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ and the explicit sieving procedure given in [3]. However, with the meromorphic continuation of $Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ already established in Proposition 3.1, we shall compute the residue more directly.

We begin with the identity

$$Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1) = L_{S_{fin}}(s, \pi) \sum_{\substack{d_0 \in \mathcal{K} \\ d_0 \text{ sq. free}}} \frac{L^S(s, \pi \otimes \chi_{d_0})}{d_0^w} \tilde{\chi}_r(d_0).$$

Interchanging the order of summation, we rewrite the sum as

$$\sum_{(n, M)=1} \frac{c_n}{n^s} \sum_{\substack{d \in \mathcal{K} \\ d \text{ sq. free}}} \frac{\chi_d(n) \tilde{\chi}_r(d)}{d^w}.$$

Set $n = n_0 n_1^2 n_2^2$ where n_0 squarefree and $p|n_1 \implies p|n_0$. We concentrate on the inner sum:

$$\begin{aligned} \sum_{d \in \mathcal{K}, sq.free} \frac{\chi_d(n) \tilde{\chi}_r(d)}{d^w} &= \sum_{\substack{d \in \mathcal{K}, d sq.free \\ (d, n_2)=1}} \frac{\chi_d(n_0) \tilde{\chi}_r(d)}{|d|^w} \\ &= \sum_{\substack{d \in \mathcal{K}, sq.free \\ (d, n_2)=1}} \frac{\chi_{n_0}(d) \chi_r(d)}{|d|^w} \end{aligned}$$

by Quadratic Reciprocity, as $d \equiv 1(4)$. We again use the orthogonality (2.3) to conclude

$$\sum_{\substack{d \in \mathcal{K}, d sq.free \\ (d, n_2)=1}} \frac{\chi_{n_0}(d) \chi_r(d)}{|d|^w} = \frac{1}{|\hat{R}|} \sum_{\psi \in \hat{R}} L_{Mn_2}^b(w, \chi_{n_0} \psi \chi_r)$$

where

$$L_b^b(w, \chi) = \sum_{\substack{d sq.free \\ (d, b)=1}} \frac{\chi(d)}{|d|^w}.$$

The residue of these L -functions is easily computed:

Lemma 4.1. *Let χ be a primitive quadratic Dirichlet character of conductor n and let $b > 0$. Then $\zeta(2w) L_b^b(w, \chi)$ can be meromorphically continued to $\Re(w) > 0$. It is analytic in this region unless $n = 1$ when it has exactly one simple pole at $w = 1$ with residue*

$$\operatorname{Res}_{w=1} \zeta(2w) L_b^b(w, \chi) = \prod_{p|b} \left(1 + \frac{1}{p}\right)^{-1}.$$

Since n_0 and r are relatively prime to M , the L -function $L_{Mn_2}^b(w, \chi_{n_0} \psi \chi_r)$ will have a pole at $w = 1$ iff $\psi = 1$ and $n = r^{2k+1} n_2^2$, $(r, n_2) = 1$.

Therefore, using Lemma 4.1,

$$\begin{aligned} \operatorname{Res}_{w=1} \left[\zeta(2w) \sum_{(n, M)=1} \frac{c_n}{n^s} L_{Mn_2}^b(w, \chi_{n_0} \chi_r) \right] \\ &= \sum_{(n_2, Mr)=1} \sum_{k=0}^{\infty} \frac{c_{r^{2k+1} n_2^2}}{(r^{2k+1} n_2^2)^s} \prod_{p|Mn_2} \left(1 + \frac{1}{p}\right)^{-1} \\ &= \left[\sum_{k=0}^{\infty} \frac{c_{r^{2k+1}}}{r^{(2k+1)s}} \right] \left[\sum_{(n_2, Mr)=1} \frac{c_{n_2^2}}{n_2^{2s}} \prod_{p|n_2} \left(1 + \frac{1}{p}\right)^{-1} \right] \prod_{p|M} \left(1 + \frac{1}{p}\right)^{-1} \\ &= \left(1 + \frac{1}{r}\right) \frac{L_{1,r}(s)}{\frac{1}{r} + L_{2,r}(s)} \prod_{p|M} \left(1 + \frac{1}{p}\right)^{-1} \sum_{(n_2, M)=1} \frac{c_{n_2^2}}{n_2^{2s}} \prod_{p|n_2} \left(1 + \frac{1}{p}\right)^{-1} \end{aligned}$$

We refer the reader to Lemma 2.3 for the explicit evaluations of $L_{1,r}$ and $L_{2,r}$.

Putting everything together, when r is a prime, $(r, M) = 1$,

$$\operatorname{Res}_{w=1} Z^b(s, w, \pi; \tilde{\chi}_r \delta_{\mathcal{K}}, 1) = R_1(s; \pi) \cdot R_r(s; \pi),$$

where

$$R_1(s; \pi) = L_{S_{fin}}(s, \pi) \frac{1}{|\hat{R}|} \prod_{p|M} \left(1 + \frac{1}{p}\right)^{-1} \sum_{(n_2, M)=1} \frac{c_{n_2^2}}{n_2^{2s}} \prod_{p|n_2} \left(1 + \frac{1}{p}\right)^{-1},$$

and

$$(4.1) \quad R_r(s; \pi) = \left(1 + \frac{1}{r}\right) \frac{L_{1r}(s)}{\frac{1}{r} + L_{2,r}(s)}.$$

For $r = 1$ a similar argument yields

$$\operatorname{Res}_{w=1} Z^b(s, w, \pi; \delta_K, 1) = R_1(s; \pi).$$

Next we note that $R_1(s; \pi)$ is well-approximated by the symmetric square L -function for $\Re(s) \geq 1/2$.

Proposition 4.2. *We have*

$$R_1(s; \pi) = L^S(2s, \pi, \operatorname{sym}^2) B_M(s; \pi)$$

where $B_M(s; \pi)$ is an absolutely convergent Euler product for $\Re(s) > 9/20$.

Proof. The Dirichlet series in the definition of $R_1(s; \pi)$ has the Euler product representation

$$\sum_{(n, M)=1} \frac{c_{n^2}}{n^{2s}} \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} = \prod_{(p, M)=1} Q_p(2s),$$

where

$$\begin{aligned} Q_p(2s) &= 1 + \frac{p}{p+1} \sum_{k \geq 1} \frac{c_{p^{2k}}}{p^{2ks}} \\ &= \sum_{k \geq 0} \frac{c_{p^{2k}}}{p^{2ks}} - \frac{1}{p+1} \sum_{k \geq 1} \frac{c_{p^{2k}}}{p^{2ks}} \\ &= L_{2,p}(s) + O(p^{-(2s+1/5-\epsilon)}). \end{aligned}$$

On the other hand the symmetric square L -function has an Euler product

$$L^S(s, \pi, \operatorname{sym}^2) = \prod_{(p, M)=1} T_p(s),$$

where

$$\begin{aligned} T_p(s) &= \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1} \left(1 - \frac{\gamma_p^2}{p^s}\right)^{-1} \\ &\quad \left(1 - \frac{\alpha_p \beta_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p \gamma_p}{p^s}\right)^{-1} \left(1 - \frac{\gamma_p \alpha_p}{p^s}\right)^{-1} \end{aligned}$$

Thus the quotient

$$\begin{aligned} \frac{L_{2,p}(s)}{T_p(2s)} &= \left(1 + \frac{c_p}{p^{2s}}\right) \left(1 - \frac{\alpha_p \beta_p}{p^{2s}}\right) \left(1 - \frac{\beta_p \gamma_p}{p^{2s}}\right) \left(1 - \frac{\gamma_p \alpha_p}{p^{2s}}\right) \\ &= 1 + O(p^{4/5-4s}). \end{aligned}$$

For the final bound we have again used the fact that, for π self-contragredient,

$$c_p = \alpha_p + \beta_p + \gamma_p = \alpha_p \beta_p + \alpha_p \gamma_p + \beta_p \gamma_p.$$

Now, an infinite product

$$\prod_p \left(1 + O(p^{4/5-4s})\right)$$

will converge absolutely provided $\frac{4}{5} - 4\Re(s) < -1$, i.e. provided $\Re(s) > 9/20$. Thus

$$B_M(s; \pi) = \prod_{(p,M)=1} \frac{L_{2,p}(s)}{T_p(2s)}$$

converges absolutely in this range. \square

To proceed further we now use the assumption that $L^S(2s, \pi, \text{sym}^2)$ has a simple pole at $s = 1/2$. By virtue of the previous proposition, $R_1(s, \pi)$ has a simple pole at $s = \frac{1}{2}$ as well. We set

$$C_M(\pi) = \lim_{s \rightarrow 1/2} (s - 1/2)R_1(s; \pi).$$

As noted at the start of this section, the double Dirichlet series $Z^\flat(s, w, \pi; \tilde{\chi}_r \delta_K, 1)$ is analytic in a neighborhood of the point $(s, w) = (1/2, 1)$ except for the two polar lines

$$w = 1 \text{ and } w + 3s = 5/2.$$

Hence near the point $(1/2, 1)$ we have the expansion

$$\begin{aligned} Z^\flat(s, w, \pi; \tilde{\chi}_r \delta_K, 1) &= \frac{A_0}{(w-1)(s-1/2)} + \frac{A_1(s)}{w-1} \\ (4.2) \quad &+ \frac{A'_0}{(w+3s-5/2)(s-1/2)} + \frac{A'_1(s)}{(w+3s-5/2)} + H(s, w), \end{aligned}$$

where

$$\begin{aligned} A_0 &= \lim_{s \rightarrow 1/2} \lim_{w \rightarrow 1} (s - 1/2)(w - 1)Z^\flat(s, w, \pi; \tilde{\chi}_r \delta_K, 1) \\ A'_0 &= \lim_{s \rightarrow 1/2} \lim_{w \rightarrow 5/2-3s} (s - 1/2)(w + 3s - 5/2)Z^\flat(s, w, \pi; \tilde{\chi}_r \delta_K, 1), \end{aligned}$$

and $A_1(s), A'_1(s)$ and $H(s, w)$ are analytic near $(1/2, 1)$.

Fix $w > 1$ and let $s \rightarrow 1/2$ in (4.2). The limit on the left hand side exists, therefore we conclude that $A_0 = -A'_0$. Hence at $s = 1/2$,

$$(4.3) \quad Z^\flat(1/2, w, \pi; \tilde{\chi}_r \delta_K, 1) = \frac{3A_0}{(w-1)^2} + \frac{B_1}{w-1} + I(w),$$

for $I(w)$ an analytic function in a neighborhood of $w = 1$. In conclusion

$$(4.4) \quad A_0 = A_0(r; \pi) = \begin{cases} C_M(\pi) & \text{if } r = 1 \\ C_M(\pi)R_r(1/2; \pi) & \text{otherwise.} \end{cases}$$

We note that Lemma 2.3 and (4.1) ensure that $R_r(s; \pi)$ makes sense for $\Re(s) > 9/20$.

5. PROOF OF THEOREM 1.1

We are finally in a position to prove our main result. We assume π_1 and π_2 are two self-contragredient cuspidal automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$ of trivial character and levels N_1, N_2 respectively. We suppose further that $L(s, \pi_1, \text{sym}^2)$ has a simple pole at $s = 1$. Choose the finite set S to contain 2, the archimedean place, and all the places of bad ramification of π_1 and π_2 .

We assume there exists a nonzero constant κ such that

$$L(1/2, \pi_1 \otimes \chi_d) = \kappa L(1/2, \pi_2 \otimes \chi_d)$$

for all positive squarefree integers $d \in \mathcal{K}$. Let $Z^b(s, w, \pi_1; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ and $Z^b(s, w, \pi_2; \tilde{\chi}_r \delta_{\mathcal{K}}, 1)$ be the associated double Dirichlet series. Letting $r = 1$, we see

$$(w-1)^2 Z^b(1/2, w, \pi_1; \delta_{\mathcal{K}}, 1) = 3C_M(\pi_1) + O(w-1).$$

by (4.3) and (4.4). On the other hand,

$$(w-1)^2 Z^b(1/2, w, \pi_1; \delta_{\mathcal{K}}, 1) = \kappa (w-1)^2 Z^b(1/2, w, \pi_2; \delta_{\mathcal{K}}, 1) = 3\kappa C_M(\pi_2) + O(w-1),$$

whence

$$C_M(\pi_1) = \kappa C_M(\pi_2).$$

Similarly, if r is prime and $(r, M) = 1$,

$$\begin{aligned} 3C_M(\pi_1)R_r(1/2; \pi_1) + O(w-1) &= (w-1)^2 Z^b(1/2, w, \pi_1; \tilde{\chi}_r \delta_{\mathcal{K}}, 1) \\ &= \kappa (w-1)^2 Z^b(1/2, w, \pi_2; \tilde{\chi}_r \delta_{\mathcal{K}}, 1) \\ &= 3\kappa C_M(\pi_2)R_r(1/2; \pi_2) + O(w-1), \end{aligned}$$

as $w \rightarrow 1$. Equivalently

$$R_r(1/2; \pi_1) = R_r(1/2; \pi_2).$$

Lemma 5.1. *There exists a rational function $h_r(t)$ such that*

$$h_r(c_r(\pi_1)) = R_r(1/2; \pi_1).$$

Moreover, the function $h_r(t)$ is monotone for r sufficiently large and $|t| < r^{1-\epsilon}$, for any $\epsilon > 0$.

Proof. Combining Lemma 2.3 with Eq. (4.1) and Eq. (2.5), we get

$$R_r(1/2; \pi_1) = (r+1)r^{3/2} \frac{rc_r(\pi_1) + 1}{c_r(\pi_1)^2(r-r^2) + c_r(\pi_1)(r^3 + 2r^2 - 2r) + (r^4 + r^3 - 1)}.$$

Let $h_r(t)$ be the function defined by

$$h_r(t) := (r+1)r^{3/2} \frac{rt + 1}{t^2(r-r^2) + t(r^3 + 2r^2 - 2r) + (r^4 + r^3 - 1)}.$$

Note that

$$h'_r(t) = \frac{r^{5/2} (r^2 - 1) (-1 + r + 2r^2 + r^3 + 2t + rt^2)}{(-1 + r^3 + r^4 - 2rt + 2r^2t + r^3t + rt^2 - r^2t^2)^2}.$$

Fix $\epsilon > 0$. If r is sufficiently large in terms of ϵ , the derivative is strictly positive for $|t| < r^{1-\epsilon}$. Thus $h_r(t)$ is monotone for t in this range. \square

To complete the proof of Theorem 1.1, we use Lemma 5.1 to conclude that if $R_r(1/2; \pi_1) = R_r(1/2; \pi_2)$, then $c_r(\pi_1) = c_r(\pi_2)$ for all r sufficiently large. The strong multiplicity one theorem now implies that $\pi_1 \simeq \pi_2$.

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