# MEAN VALUES OF BIQUADRATIC ZETA FUNCTIONS

GAUTAM CHINTA

## 1. Introduction

Let $h(d)$ be the number of primitive inequivalent binary quadratic forms of discriminant $d$. In Disquisitiones §302 and 304 [8], Gauss gives conjectures for the average value of $h(d)$ if $d < 0$, and for the average value of $h(d)\log(\epsilon_d)$ if $d > 0$, where in the latter case, $\epsilon_d$ is derived from the fundamental solution to Pell's equation. Gauss's conjectures were first proved by Lipschitz [12] (for negative discriminants) and Siegel (for positive discriminants) [14].

When $d$ is a fundamental discriminant and $\chi_d$ the primitive real Dirichlet character associated to the quadratic extension $\mathbb{Q}[\sqrt{d}]$, Dirichlet's class number formula relates the value of the $L$-series of $\chi_d$ at $s = 1$ to the class number $h(d)$. Thus Gauss's conjectured asymptotics are equivalent to conjectures for sums of the type

$$\sum_{0<\pm d<X} L(1, \chi_d)$$

as $X \to \infty$. Subsequent authors have investigated asymptotics for sums of the type

$$\sum_{0<\pm d<X} L(s, \chi_d)$$

for $\mathrm{Re}(s) \geq 1/2$.

The greatest difficulties occur at the point $s = 1/2$. Jutila [11], verifying a conjecture of Goldfeld and Viola [10], showed that

$$\sum_{0<\pm d<X} L(1/2, \chi_d) \sim c_1 X \log X + c_2 x + O(x^{3/4+\epsilon}) \tag{1.1}$$

for certain constants $c_1, c_2$. Takhtadzjan and Vinogradov [15] establish a similar result. Using the theory of metaplectic Eisenstein series, Goldfeld and Hoffstein [9] improved the exponent on Jutila's error term to $19/32+\epsilon$. They exploited the fact—first noticed by Siegel [13]—that the Dirichlet $L$-function appears in the Fourier expansion of the half-integral weight Eisenstein series on $\Gamma_0(4)$. By taking the Mellin transform of this Eisenstein series, one obtains a double Dirichlet series whose coefficients involve the $L$-functions $L(s, \chi_d)$. The asymptotic of [9] then follows from standard Tauberian techniques. This approach has been vastly generalized in recent years. We refer the reader to [3] for a survey of the role of metaplectic Eisenstein series in constructing multiple Dirichlet series and applications to number theory and the theory of automorphic forms.

As $\zeta(s)L(s,\chi_d)$ is the zeta function of the field $\mathbb{Q}[\sqrt{d}]$, the result (1.1) can be viewed as an asymptotic for central values of zeta functions of quadratic extensions of $\mathbb{Q}$. In this paper we prove an asymptotic formula for a weighted sum of central values of zeta functions of biquadratic extensions of a number field $K$. To keep notation to a minimum in this introduction, we content ourselves with stating our main result (Theorem 2.3) over the base field $\mathbb{Q}$.

We recall that if $L = \mathbb{Q}[\sqrt{d_1}, \sqrt{d_2}]$ with $d_1, d_2$ fundamental discriminants, $(d_1, d_2) = 1$, the zeta function of the field $L$ is

$$\zeta_L(s) = \zeta(s)L(s,\chi_{d_1})L(s,\chi_{d_2})L(s,\chi_{d_1d_2}),$$

where $\zeta = \zeta_{\mathbb{Q}}$ is the Riemann zeta function. When $d$ is not a fundamental discriminant, we continue to let $\chi_d$ denote the quadratic character associated to the extension $\mathbb{Q}[\sqrt{d}]$ of $\mathbb{Q}$. Let $L_2(s,\chi_d)$ denote the $L$-function with the Euler factor at the prime 2 removed. Let $f$ be a smooth, compactly supported test function satisfying $\int_0^\infty f(x)dx = 1$. Our main result is

$$\sum_{d_1,d_2 \text{ odd}} a(d_1,d_2)L_2(\tfrac{1}{2},\chi_{d_1})L_2(\tfrac{1}{2},\chi_{d_2})L_2(\tfrac{1}{2},\chi_{d_1d_2})f\left(\frac{d_1d_2}{X}\right) \sim \frac{\zeta_2(\frac{3}{2})\zeta_2(2)^3}{2\cdot 4!}X\log^4 X, \tag{1.2}$$

as $X \to \infty$. For lower order terms in the asymptotic and an error term, see Theorem 2.3. The weighting factor $a(d_1,d_2)$ satisfies

- $a(d_1,d_2) = 1$ if $d_1d_2$ squarefree
- The weights are "small" in the sense that, for $d_1d_2$ squarefree,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2w}} \left( \sum_{m_1m_2=n^2} a(m_1d_1, m_2d_2) \right)$$

  converges absolutely for $\mathrm{Re}(w) > 1/2$.

The weighting factor is described in greater detail in Section 4.1. It is natural in the sense that its presence is required in order that a certain multiple Dirichlet series have a full group of functional equations. According to a conjecture of D. Bump, the multiple Dirichlet series we construct should coincide with a Whittaker coefficient of a metaplectic Eisenstein series on the double cover of $GL_6$. See section 3 for further remarks.

**Acknowledgments.** It is a great pleasure to thank Prof. J. Hoffstein for numerous enlightening conversations. I am also grateful to the referee for suggesting several improvements to this paper.

## 2. Preliminaries

Let $K$ be a number field with ring of integers $\mathcal{O}$. Let $S_f$ be a finite set of non-archimedean places such that $S_f$ contains all places dividing 2 and the ring of $S_f$-integers $\mathcal{O}_{S_f}$ has class number 1. Let $S_\infty$ denote the set of archimedean places and let $S = S_f \cup S_\infty$.

Let $\left(\frac{a}{*}\right)$ be the quadratic residue symbol attached to the extension $K(\sqrt{a})$ of $K$. We extend this symbol as in Fisher and Friedberg [6]. We review the definition.

For each place $v$, let $K_v$ denote the completion of $K$ at $v$. For $v$ nonarchimedean, let $P_v$ denote the corresponding ideal of $\mathcal{O}$, and let $q_v = |P_v|$ denote its norm. Let $C = \prod_{v\in S_f} P_v^{n_v}$ with $n_v = \max\{\mathrm{ord}_v(4), 1\}$. Let $H_C$ be the narrow ray class group modulo $C$ and let $R_C = H_C \otimes \mathbb{Z}/2\mathbb{Z}$. Write the finite group $R_C$ as a direct product of cyclic groups, choose a generator for each, and let $\mathcal{E}_0$ be a set of ideals of $\mathcal{O}$ prime to $S$ which represent these generators. For each $E_0 \in \mathcal{E}_0$ choose $m_{E_0} \in K^\times$ such that $E_0\mathcal{O}_{S_f} = m_{E_0}\mathcal{O}_{S_f}$. Let $\mathcal{E}$ be a full set of representatives for $R_C$ of the form $\prod_{E_0\in\mathcal{E}_0} E_0^{n_{E_0}}$, with $n_{E_0} \in \mathbb{Z}$. If $E = \prod_{E_0\in\mathcal{E}_0} E_0^{n_{E_0}}$ is such a representative, then let $m_E = \prod_{E_0\in\mathcal{E}_0} m_{E_0}^{n_{E_0}}$. Note that $E\mathcal{O}_{S_f} = m_E\mathcal{O}_{S_f}$ for all $E \in \mathcal{E}$. For convenience we suppose that $\mathcal{O} \in \mathcal{E}$ and $m_{\mathcal{O}} = 1$.

Let $\mathcal{J}(S)$ denote the group of fractional ideals of $\mathcal{O}$ coprime to $S_f$. Let $I, J \in \mathcal{J}(S)$ be coprime. Write $I = (m)EG^2$ with $E \in \mathcal{E}$, $m \in K^\times$, $m \equiv 1 \bmod C$, and $G \in \mathcal{J}(S)$ such that $(G, J) = 1$. Then as in [6], the quadratic residue symbol $\left(\frac{mm_E}{J}\right)$ is defined, and if $I = (m')E'G'^2$ is another such decomposition, then $E' = E$ and $\left(\frac{m'm_E}{J}\right) = \left(\frac{mm_E}{J}\right)$. In view of this define the quadratic residue symbol $\left(\frac{I}{J}\right)$ by $\left(\frac{I}{J}\right) = \left(\frac{mm_E}{J}\right)$. For $I = I_0I_1^2$ with $I_0$ squarefree we denote by $\chi_I$ the character $\chi_I(J) = \chi_{I_0}(J) = \left(\frac{I_0}{J}\right)$. Further, in the expression $\chi_I(\hat{J})$, we let $\hat{J}$ represent the part of $J$ coprime to $I_0$. This character $\chi_I$ depends on the choices above, but we suppress this from the notation.

**Proposition 2.1** (Reciprocity)**.** [6] *Let $I$, $J \in \mathcal{J}(S)$ be coprime, and $\alpha(I,J) = \chi_I(J)\chi_J(I)^{-1}$. Then $\alpha(I,J)$ depends only on the images of $I$ and $J$ in $R_C$.*

Let $\mathcal{I}(S)$ denote the set of integral ideals prime to $S_f$. Let $L_S(s,\chi_J)$ be the $L$-function of the character $\chi_J$, with the places in $S$ removed. If $\xi$ is any idèle class character then the $L$-function $L(s,\xi)$ satisfies a functional equation

$$L_\infty(s,\xi)L(s,\xi) = \epsilon(s,\xi)L_\infty(1-s,\xi)\,L(1-s,\xi^{-1}), \tag{2.1}$$

where $\epsilon(s,\xi)$ is the epsilon factor of $\xi$ and $L_\infty(s,\xi)$ is the archimedean component of the $L$-function.

**Proposition 2.2.** *Let $E, J \in \mathcal{O}(S)$ be squarefree with associated characters $\chi_E, \chi_J$ of conductors $\mathfrak{f}_E, \mathfrak{f}_J$ respectively. Suppose that $\chi_J = \chi_E\chi_I$ with $I \in K^\times$, $I \equiv 1 \bmod C$. Let $\psi$ be another character unramified outside S. Then*

$$\epsilon(s,\chi_J\psi) = \epsilon(1/2,\chi_I)\psi(|\mathfrak{f}_J/\mathfrak{f}_E)\,(|\mathfrak{f}_J/\mathfrak{f}_E|)^{1/2-s}\,\epsilon(s,\chi_E\psi). \tag{2.2}$$

Here $\epsilon(1/2, \chi_I)$ is given by a (normalized) Gauss sum, as in Tate's thesis. We may now state our main result.

**Theorem 2.3.** *Let $a(I_2, I_4)$ be the weighting factors given by (4.12). Let $f$ be a smooth, compactly supported test function on $(0, \infty)$, satisfying*

$$\int_0^\infty f(x)dx = 1.$$

*Then for any $\epsilon > 0$, as $X \to \infty$,*

$$\sum_{I_2, I_4 \in \mathcal{I}(S)} a(I_2, I_4) L_S(\tfrac{1}{2}, \chi_{I_2}) L_S(\tfrac{1}{2}, \chi_{I_4}) L_S(\tfrac{1}{2}, \chi_{I_2 I_4}) f\left(\tfrac{|I_2 I_4|}{X}\right) \tag{2.3}$$

$$\sim \frac{\zeta_S(\frac{3}{2})\zeta_S(2)^3}{4!} \prod_{P \in S}\left(1 - \frac{1}{|P|}\right) X \log^4 X + \sum_{i=0}^{3} A_i X (\log X)^i + O(X^{3/4+\epsilon})$$

*where $\zeta$ denotes the zeta function of $K$ and the constants $A_0, A_1, A_2, A_3$ are all effectively computable in terms of the Mellin transform of the test function $f$. The implicit constant in the error term depends on $\epsilon, K$ and $S$.*

The proof of the theorem is a consequence of the analytic continuation of a certain multiple Dirichlet series constructed in Section 4.

## 3. Dynkin diagrams and multiple Dirichlet series

In 1996 Bump suggested a correspondence between quadratic multiple Dirichlet series and Dynkin diagrams. Suppose that we are given a simply-laced Dynkin diagram, with vertices $v_1, \cdots, v_r$. We can try to attach to the Dynkin diagram a multiple Dirichlet series which is roughly of the form:

$$\sum_{n_1, n_2, \ldots, n_r = 1}^{\infty} \left[ \prod_{j > i,\ v_j \text{ adjacent to } v_i} \left(\frac{n_i}{n_j}\right)\right] n_1^{-s_1} \cdots n_r^{-s_r}.$$

When $n_i n_j$ is not squarefree the symbol $\left(\frac{n_i}{n_j}\right)$ must replaced by an appropriate weighting factor. Summing over $n_i$, while fixing all $n_k$ with $k \neq i$, will produce a function of $s_i$. Bump suggested that weighting factors could be chosen in such a way that this function would satisfy a natural functional equation as $s_i \mapsto 1 - s_i$. This functional equation will induce (in the multiple Dirichlet series) a linear change of the variables $s_1, s_2, \ldots, s_r$ sending $s_i \to 1 - s_i$, and $s_j \to s_j + s_i - 1/2$ if $v_j$ is adjacent to $v_i$. The other $s_k$ are left unchanged. Denoting this functional equation by $\sigma_i$, we have the relations

$$\sigma_i^2 = 1, \qquad (\sigma_i \sigma_j)^{\epsilon(i,j)} = 1,$$

where $\epsilon(i, j) = 3$ if $v_i$ and $v_j$ are adjacent nodes in the Dynkin diagram, and $\epsilon(i, j) = 2$ if they are not, see [3]. These are the well-known Coxeter relations generating the Weyl group associated with the Dynkin diagram. By simple one-variable convexity estimates, the region of absolute convergence of the multiple Dirchlet series contains the complement of a bounded subset of a Weyl chamber. Since the Weyl group acts transitively on Weyl chambers, it should therefore be possible (provided the appropriate weighting factors can be constructed) to analytically continue the multiple Dirichlet series to the complement of a bounded subset of $\mathbb{C}^r$, and hence—by the convexity principle of several complex variables—to all of $\mathbb{C}^r$. If this can be done, the multiple Dirichlet series is said to be "perfect."

This procedure has been carried out in detail for the Dynkin diagrams $A_2, A_3$ and $D_4$, see [14],[9], [6], [5], [4]. These multiple Dirichlet series give asymptotics for mean values of quadratic twists of $L$-functions on $GL(1), GL(2)$ and $GL(3)$. Bump conjectures that the multiple Dirichlet series constructed in this manner coincide with the Whittaker coefficients of Eisenstein series on the metaplectic double cover of the split simply connected semisimple group associated with this Dynkin diagram. In an unpublished computation of Bump and Hoffstein, this has been verified when the Cartan type of the Dynkin diagram is $A_2$, in which case this Dirichlet series is associated with the Eisenstein series on the metaplectic double cover of $SL(3)$.

There is also a relation between Dynkin diagrams and multiple Dirichlet series constructed from higher order twists. This construction together with a conjectural connection to Fourier coefficients of Eisenstein series on $n$-fold metaplectic covers will be described in [2]. Brubaker and Bump have recently made great progress on showing that these multiple Dirichlet series are perfect, see [1].

In the following section we construct the multiple Dirichlet series associated to the Dynkin diagram $A_5$. Theorem 2.3 will be a consequence of the analytic properties of this series.

We conclude this section with an observation by the referee which lends further credence to the conjecture above. The construction of the weighting factors alluded to above turns out to be equivalent to the construction of a rational function invariant under a certain action of the Weyl group. This action is described in the following section and the rational function is presented in the Appendix. Multiplying the denominator of the rational function by $(1+x)(1+y)(1+z)(1+w)(1+v)$ allows us to pull 15 zeta functions out of the multiple Dirichlet series. These 15 factors coincide (for a suitable variable choice) with the normalizing zeta factor of the Eisenstein series on the metaplectic double cover of $GL(6)$.

## 4. A multiple Dirichlet series associated to the Dynkin diagram $A_5$

We retain the notation of section 2. Let $\psi_1, \ldots, \psi_5$ be quadratic idèle class characters unramified outside $S$. Define the the multiple Dirichlet series

$$Z_S(s_1, s_2, s_3, s_4, s_5; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5) = \sum_{I_1, \ldots, I_5 \in \mathcal{I}(S)} \frac{\chi_{I_2}(\hat{I}_1)\chi_{I_2}(\hat{I}_3)\chi_{I_4}(\hat{I}_3)\chi_{I_4}(\hat{I}_5) \prod_{i=1}^{i=5} \psi_i(I_i)}{\prod_{i=1}^{i=5} |I_i|^{s_i}} g(I_1, I_2, I_3, I_4, I_5), \tag{4.1}$$

where $g$ is a certain weighting factor to be specified below. This factor must be included to insure that $Z_S$ satisfies the proper group of functional equations.

### 4.1. Construction of the weighting factor.

We begin by listing the properties we need $g$ to have. Then we show that such a $g$ exists. Firstly, we require that $g$ is multiplicative in the following sense

$$g(I_1, I_2, I_3, I_4, I_5) = \prod_{P^{\alpha_i} || I_i} g(P^{\alpha_1}, P^{\alpha_2}, P^{\alpha_3}, P^{\alpha_4}, P^{\alpha_5}) \tag{4.2}$$

and that

$$g(I_1, I_2, I_3, I_4, I_5) = g(I_5, I_4, I_3, I_2, I_1). \tag{4.3}$$

Moreover, $g$ must be chosen so that (4.1) satisfies functional equations as $s_i \mapsto 1 - s_i$. Precisely, we require

- For $I_2 \in \mathcal{I}(S)$ set $I_2 = J_0 J_1^2$ with $J_0, J_1 \in \mathcal{I}(S)$, $J_0$ squarefree. Then, for fixed $I_3, I_4, I_5 \in \mathcal{I}(S)$,

$$\sum_{I_1} \frac{\chi_{J_0}(\hat{I}_1)\psi_1(I_1)}{|I_1|^{s_1}} g(I_1, I_2, I_3, I_4, I_5) = L(s_1, \chi_{J_0}\psi_1) Q^{(1)}_{I_2, I_3, I_4, I_5}(s_1, \psi_1), \tag{4.4}$$

  where the weighting polynomial $Q^{(1)}$ satisfies the functional equation

$$Q^{(1)}_{I_2, I_3, I_4, I_5}(s_1, \psi_1) = |J_1|^{1-2s_1} Q^{(1)}_{I_2, I_3, I_4, I_5}(1 - s_1, \psi_1). \tag{4.5}$$

- For $I_1, I_3 \in \mathcal{I}(S)$ set $I_1 I_3 = J_0 J_1^2$ with $J_0, J_1 \in \mathcal{I}(S)$, $J_0$ squarefree. Then, for fixed $I_4, I_5 \in \mathcal{I}(S)$,

$$\sum_{I_2} \frac{\chi_{J_0}(\hat{I}_2)\psi_2(I_2)}{|I_2|^{s_2}} g(I_1, I_2, I_3, I_4, I_5) = L(s_2, \chi_{J_0}\psi_2) Q^{(2)}_{I_1, I_3, I_4, I_5}(s_2, \psi_2), \tag{4.6}$$

  where the weighting polynomial $Q^{(2)}$ satisfies the functional equation

$$Q^{(2)}_{I_1, I_3, I_4, I_5}(s_2) = |J_1|^{1-2s_2} Q^{(2)}_{I_1, I_3, I_4, I_5}(1 - s_2, \psi_2). \tag{4.7}$$

- For $I_2, I_4 \in \mathcal{I}(S)$ set $I_2 I_4 = J_0 J_1^2$ with $J_0, J_1 \in \mathcal{I}(S)$, $J_0$ squarefree. Then, for fixed $I_1, I_5 \in \mathcal{I}(S)$,

$$\sum_{I_3} \frac{\chi_{J_0}(\hat{I}_3)\psi_3(I_3)}{|I_3|^{s_3}} g(I_1, I_2, I_3, I_4, I_5) = L(s_3, \chi_{J_0}) Q^{(3)}_{I_1,I_2,I_4,I_5}(s_3, \psi_3), \tag{4.8}$$

where the weighting polynomial $Q^{(3)}$ satisfies the functional equation

$$Q^{(3)}_{I_1,I_2,I_4,I_5}(s_3) = |J_1|^{1-2s_3} Q^{(3)}_{I_1,I_2,I_4,I_5}(1 - s_3, \psi_3). \tag{4.9}$$

The symmetry (4.3) implies that sums over $I_4$ and $I_5$ will satisfy analogous functional equations in $s_4$ and $s_5$. We also note that as $g$ is multiplicative, the Dirichlet polynomials $Q^{(j)}$ will be (finite) Euler products for $1 \leq j \leq 5$.

We now show that a weighting function with the above properties exists. By multiplicativity it suffices to define $g(P^j, P^k, P^l, P^m, P^n)$ for $j, k, l, m, n \geq 0$ and $P$ a prime ideal of norm $p$. We introduce the new variables

$$x = p^{-s_1}, y = p^{-s_2}, z = p^{-s_3}, w = p^{-s_4}, v = p^{-s_5}$$

and consider the rational function

$$H(x, y, z, w, v)$$

given in the appendix. The proposition below is readily verified using any computer algebra system. We hope in a later work to give a more conceptual and less computational construction of the rational functions needed to construct multiple Dirichlet series associated to a general simply-laced Dynkin diagram, as explained in the previous section.

**Proposition 4.1.** *The rational function $H$ satisfies*

- $H(x, y, z, w, v) = H(v, w, z, y, x)$
- *The functions*

$$(1 - x)\Big[H(x, y, z, w, v) + H(x, -y, z, w, v)\Big]$$

*and*

$$\frac{1}{x\sqrt{p}}\Big[H(x, y, z, w, v) - H(x, -y, z, w, v)\Big]$$

*are invariant under*

$$(x, y, z, w, v) \mapsto (\tfrac{1}{px}, xy\sqrt{p}, z, w, v).$$

- *The functions*

$$(1 - y)\Big[H(x, y, z, w, v) + H(-x, y, -z, w, v)\Big]$$

*and*

$$\frac{1}{y\sqrt{p}}\Big[H(x, y, z, w, v) - H(-x, y, -z, w, v)\Big]$$

*are invariant under*

$$(x, y, z, w, v) \mapsto (xy\sqrt{p}, \tfrac{1}{py}, yz\sqrt{p}, w, v).$$

- *The functions*

$$(1-z)\Big[H(x,y,z,w,v)+H(x,-y,z,-w,v)\Big]$$

*and*

$$\frac{1}{z\sqrt{p}}\Big[H(x,y,z,w,v)-H(x,-y,z,-w,v)\Big]$$

*are invariant under*

$$(x,y,z,w,v)\mapsto(x,yz\sqrt{p},\tfrac{1}{pz},wz\sqrt{p},v).$$

If we define $g(P^j,P^k,P^l,P^m,P^n)$ by

$$H(x,y,z,w,v)=\sum_{j,k,l,m,n\geq 0} g(P^j,P^k,P^l,P^m,P^n)x^jy^kz^lw^mv^n \tag{4.10}$$

and extend multiplicatively, then, in view of the previous proposition, these coefficients $g$ will satisfy (4.2) - (4.8), as desired.

We can now also describe the weighting factors $a(I_1,I_2)$ that appear in the statement of our main result, Theorem 2.3. Define $\mathcal{P}_{I_2,I_4}(s_1,s_3,s_5;P)$ to be equal to

$$\begin{cases} 1-\frac{\chi_{I_2}(P)}{|P|^{s_1}} & \text{if } \mathrm{ord}_P(I_2) \text{ even, } \mathrm{ord}_P(I_4) \text{ odd} \\ 1-\frac{\chi_{I_4}(P)}{|P|^{s_5}} & \text{if } \mathrm{ord}_P(I_2) \text{ odd, } \mathrm{ord}_P(I_4) \text{ even} \\ 1-\frac{\chi_{I_2I_4}(P)}{|P|^{s_3}} & \text{if } \mathrm{ord}_P(I_2) \text{ odd, } \mathrm{ord}_P(I_4) \text{ odd} \\ \left(1-\frac{\chi_{I_2}(P)}{|P|^{s_1}}\right)\left(1-\frac{\chi_{I_4}(P)}{|P|^{s_5}}\right)\left(1-\frac{\chi_{I_2I_4}(P)}{|P|^{s_3}}\right) & \text{otherwise} \end{cases} \tag{4.11}$$

Then the weighting factor $a(I_2,I_4)$ of Theorem 2.3 is given by

$$\prod_{\substack{P^{\alpha_2}||I_2\\ P^{\alpha_4}||I_4}} \mathcal{P}_{I_2,I_4}(\tfrac12,\tfrac12,\tfrac12;P)\left(\sum_{j,l,n\geq 0}\frac{\chi_{I_2}(\hat{P}^j)\chi_{I_2}(\hat{P}^l)\chi_{I_4}(\hat{P}^n)\chi_{I_4}(\hat{P}^n)}{P^{j/2}P^{l/2}P^{n/2}}g(P^j,P^{\alpha_2},P^l,P^{\alpha_4},P^n)\right). \tag{4.12}$$

4.2. **Functional equations for $Z_S(s_1,s_2,s_3,s_4,s_5)$.** Summing over $I_j$ first in the series (4.1) defining $Z_S$ will produce an $L$-function in the variable $s_j$, which will have a functional equation as $s_j\mapsto 1-s_j$. This will lead to a functional equation for the multiple Dirichlet series $Z_S$. We exhibit the $s_1\mapsto 1-s_1$ relation in detail.

For $I_2\in\mathcal{I}(S)$ set $I_2=J_0J_1^2$ with $J_0,J_1\in\mathcal{I}(S),J_0$ squarefree. Let $I_2,I_3,I_4,I_5$ be fixed ideals in $\mathcal{I}(S)$. Consider the sum

$$\sum_{I_1\in\mathcal{I}(S)}\frac{\chi_{I_2}(I_1)\psi_1(I_1)}{|I_1|^{s_1}}g(I_1,I_2,I_3,I_4,I_5)=\hat{L}_S(s_1,\chi_{I_2}\psi_1),$$

say, where $\hat{L}$ denotes the product of the $L$-function with the weighting polynomial as in (4.4). Let us write $\chi_{I_2}=\chi_E\chi_{J'}$, with $J'\in K^\times, J'\equiv 1 \bmod C$. Combining the functional equation of the $L$-function (2.1) with that of the weighting polynomial (4.5) and taking care to replace and remove the primes in $S_f$, we find that $\hat{L}_S(s_1, \chi_{I_2}\psi_1)$ satisfies the functional equation

$$\begin{aligned}\hat{L}_S(s_1, \chi_{I_2}\psi_1) &= \frac{L_\infty(1-s_1, \chi_{I_2}\psi_1)}{L_\infty(s_1, \chi_{I_2}\psi_1)} \prod_{P\in S_f} \left(\frac{1-\chi_{I_2}(P)\psi_1(P)|P|^{-s_1}}{1-\chi_{I_2}(P)\psi_1(P)|P|^{s_1-1}}\right) \\ &\quad \times \left|\frac{\mathfrak{f}_{J_0}}{\mathfrak{f}_E}\right|^{1/2-s_1} |J_1^2|^{1/2-s_1}\epsilon(1-s, \chi_E\psi_1)\hat{L}_S(1-s_1, \chi_{I_2}\psi_1) \\ &= A(s_1, E, \psi_1) \prod_{P\in S_f} \left(\frac{1-\chi_{I_2}(P)\psi_1(P)|P|^{-s_1}}{1-\chi_{I_2}(P)\psi_1(P)|P|^{s_1-1}}\right) \\ &\quad \times |I_2|^{1/2-s_1}\hat{L}_S(1-s_1, \chi_{I_2}\psi_1)\end{aligned}$$

where $A(s_1, E, \psi_1)$ depends only on the class $E$ of $I_2$ in $R_C$. The same is true of the quotient of Euler factors. Thus

$$\hat{L}_S(s_1, \chi_{I_2}\psi_1) = B(s_1, E, \psi_1)|I_2|^{1/2-s_1}\hat{L}_S(1-s_1, \chi_{I_2}\psi_1) \tag{4.13}$$

We will find it convenient to extend the notation of (4.1) to allow arbitrary functions on $R_C$ in place of the $\psi_i$. In particular, if $E$ is a class in $R_C$, we let $\delta_E$ denote the characteristic function of this class and

$$Z_S(s_1, s_2, s_3, s_4, s_5; \psi_1\delta_E, \psi_2, \dots, \psi_5) = \sum_{\substack{I_1,\dots,I_5\in\mathcal{I}(S)\\ I_1\sim E}} \frac{\chi_{I_2}(I_1)\chi_{I_2}(I_3)\chi_{I_4}(I_5)\chi_{I_4}(I_3)\prod_{i=1}^{i=5}\psi_i(I_i)}{\prod_{i=1}^{i=5}|I_i|^{s_i}} g(I_1, I_2, I_3, I_4, I_5),$$

Summing (4.13) over $I_2$ projecting to $E$ in $R_C$,

$$\begin{aligned}&\prod_{P\in S_f}\left(1-\frac{\chi_{I_2}(P)\psi_1(P)}{|P|^{1-s_1}}\right) Z_S(s_1, s_2, s_3, s_4, s_5; \psi_1, \psi_2\delta_E, \psi_3, \psi_4, \psi_5) \\ &\qquad = A(s_1, E, \psi_1)\prod_{P\in S_f}\left(1-\frac{\chi_{I_2}(P)\psi_1(P)}{|P|^{s_1}}\right) \\ &\qquad\quad Z_S(1-s_1, s_2+s_1-1/2, s_3, s_4, s_5; \psi_1, \psi_2\delta_E, \psi_3, \psi_4, \psi_5),\end{aligned}$$

or equivalently,

$$\begin{aligned}&\prod_{P\in S_f}\left(1-\frac{\psi_1(P^2)}{|P|^{2-2s_1}}\right) Z_S(s_1, s_2, s_3, s_4, s_5; \psi_1, \psi_2\delta_E, \psi_3, \psi_4, \psi_5) \\ &\quad = A(s_1, E, \psi_1)\prod_{P\in S_f}\left(1-\frac{\chi_{I_2}(P)\psi_1(P)}{|P|^{s_1}}\right)\left(1+\frac{\chi_{I_2}(P)\psi_1(P)}{|P|^{2-2s_1}}\right) \\ &\qquad\quad Z_S(1-s_1, s_2+s_1-1/2, s_3, s_4, s_5; \psi_1, \psi_2\delta_E, \psi_3, \psi_4, \psi_5),\end{aligned}$$

Now summing both sides over $E \in \mathcal{E}$ will produce a functional equation for $Z_S(s_1, s_2, s_3, s_4, s_5; \psi_1, \psi_2\delta_E, \psi_3, \psi_4, \psi_5)$. Functional equations for $Z_S$ as

$s_i \mapsto 1 - s_i$ for $i = 2, 3, 4, 5$ can be established similarly. We list the results below.

**Theorem 4.2.** *Let* $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5)$. *The multiple Dirichlet series* $Z_S$ *satisfies the following functional equations:*

- *Let* $\sigma_1(\mathbf{s}) = (1 - s_1, s_1 + s_2 - 1/2, s_3, s_4, s_5)$. *Then*

$$\prod_{P \in S_f} \left(1 - \frac{\psi_1(P^2)}{|P|^{2-2s_1}}\right) Z_S(\mathbf{s}; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$$
$$= \sum_{\xi_2 \in \hat{R}_C} \Phi(s_1; \xi, \psi_1) Z_S(\sigma_1 \mathbf{s}; \psi_1, \psi_2 \xi_2, \psi_3, \psi_4, \psi_5)$$

- *Let* $\sigma_2(\mathbf{s}) = (s_1 + s_2 - 1/2, 1 - s_2, s_2 + s_3 - 1/2, s_4, s_5)$. *Then*

$$\prod_{P \in S_f} \left(1 - \frac{\psi_2(P^2)}{|P|^{2-2s_2}}\right) Z_S(\mathbf{s}; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$$
$$= \sum_{\xi_1, \xi_3 \in \hat{R}_C} \Phi(s_2; \xi_1, \xi_3, \psi_2) Z_S(\sigma_2 \mathbf{s}; \psi_1 \xi_1, \psi_2, \psi_3 \xi_3, \psi_4, \psi_5)$$

- *Let* $\sigma_3(\mathbf{s}) = (s_1, s_2 + s_3 - 1/2, 1 - s_3, s_3 + s_4 - 1/2, s_5)$. *Then*

$$\prod_{P \in S_f} \left(1 - \frac{\psi_3(P^2)}{|P|^{2-2s_3}}\right) Z_S(\mathbf{s}; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$$
$$= \sum_{\xi_2, \xi_4 \in \hat{R}_C} \Phi(s_3; \xi_2, \xi_4, \psi_3) Z_S(\sigma_3 \mathbf{s}; \psi_1, \psi_2 \xi_2, \psi_3, \psi_4 \xi_4, \psi_5)$$

- *Let* $\tau(\mathbf{s}) = (s_5, s_4, s_3, s_2, s_1)$. *Then*

$$Z_S(\mathbf{s}; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5) = Z_S(\tau \mathbf{s}; \psi_5, \psi_4, \psi_3, \psi_2, \psi_1)$$

*The functions* $\Phi_i(s; \cdot)$ *are linear combinations of quotients of Gamma functions multiplied by finite Euler products in* $s$ *and* $1 - s$.

4.3. **Analytic continuation of the multiple Dirichlet series.** We will be brief in this section as the procedure for analytically continuing a multiple Dirichlet series satisfying sufficiently many functional equations has been described in detail elsewhere, see [4],[5]. We remark only that the region

$$R = \{(s_1, s_2, s_3, s_4, s_5) : \mathrm{Re}(s_1) \geq \tfrac{1}{2}\}$$

is a fundamental domain for the action of the group $G$ generated by $\sigma_1, \sigma_2, \sigma_3$ and $\tau$ on $\mathbb{C}^5$. Using standard convexity estimates for $L$-series, the multiple Dirichlet series $Z_S(s_1, s_2, s_3, s_4, s_5; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$ can be shown to be meromorphic in the intersection of $R$ with the complement of the ball of radius 2 centered at the origin. By the Hartogs' theorem in several complex variables, it follows that $Z_S$ can be meromorphically continued to $\mathbb{C}^5$. Moreover, the polar divisor of $Z_S$ will be contained in the set of translates of the hyperplane $\{s_2 = 1\}$ by the group $G$. In particular, in the case

$\psi_i = 1, i = 1, \ldots, 5$, there exist exactly 8 polar hyperplanes passing through the point $(1/2, 1, 1/2, 1, 1/2)$.

## 5. Computation of the residue

Henceforth we specialize our investigations to the series

$$Z_S(s_1, s_2, s_3, s_4, s_5) = Z_S(s_1, s_2, s_3, s_4, s_5; 1, 1, 1, 1, 1).$$

We wish to determine the analytic behavior of $Z_S(1/2, s, 1/2, s, 1/2)$ as $s \to 1^+$. For this we will need to know some of the properties of the $A_3$ multiple Dirichlet series

$$\begin{aligned} Y_S(s_1, s_2, s_3) &= \lim_{s_4, s_5 \to \infty} Z_S(s_1, s_2, s_3, s_4, s_5) \\ &= \sum_{I_1, I_2, I_3 \in \mathcal{I}(S)} \frac{\chi_{I_2}(\hat{I}_1)\chi_{I_2}(\hat{I}_2)}{\prod_{i=1}^{i=3} |I_i|^{s_i}} g_Y(I_1, I_2, I_3). \end{aligned}$$

The multiplicative weighting coefficient $g_Y$ is determined by the generating series

$$\begin{aligned} (5.1) \quad H_Y(x, y, z) &= \sum_{j,k,l \geq 0} g(P^j, P^k, P^l) x^j y^k z^l \\ &= \frac{1 - y(x + z - xz) + py^2 xz(1 - z - x) + py^3 z^2 x^2}{(1 - x)(1 - y)(1 - z)(1 - px^2y^2)(1 - py^2z^2)(1 - p^2x^2y^2z^2)} \end{aligned}$$

as can be established by setting $v = w = 0$ in $H$.

This series was studied in [7]. Of relevance to us, is that the behavior of $Y_S(s_1, s_2, s_3)$ near the point $(1, 1, 1/2)$ is given by

$$\begin{aligned} (5.2) \quad (1/r_S)Y_S(s_1, s_2, s_3) &= \frac{\zeta_S(s_3)\zeta_S(2s_2)\zeta_S(2s_2 + 2s_3 - 1)}{s_1 - 1} \\ &+ \frac{\zeta_S(2s_1)\zeta_S(2s_3)\zeta_S(s_1 + s_3)}{s_2 - 1} + \frac{\zeta_S(2s_1)\zeta_S(2 - 2s_3)\zeta_S(s_1 - s_3 + 1)}{s_2 + s_3 - 1} \\ &+ f(s_1, s_2, s_3) \end{aligned}$$

where $f(s_1, s_2, s_3)$ is holomorphic in a neighborhood of $(1, 1, 1/2)$ and

$$r_S = \prod_{p \in S_f} \left(1 - \frac{1}{p}\right).$$

This fact may be established by methods similar to, but simpler than the methods of the following paragraph.

By the results of the previous section, $Z_S(s_1, s_2, s_3, s_4, s_5)$ has 8 polar hyperplanes going through the point $(1/2, 1, 1/2, 1, 1/2)$. These are:

$$\begin{aligned} (5.3) \quad &\{s_2 = 1\}, \{s_1 + s_2 = 3/2\}, \{s_1 + s_2 + s_3 = 2\}, \{s_2 + s_3 = 3/2\}, \\ &\{s_4 = 1\}, \{s_5 + s_4 = 3/2\}, \{s_5 + s_4 + s_3 = 2\}, \{s_4 + s_3 = 3/2\}. \end{aligned}$$

We need to compute the residue of $Z_S$ along each of these polar divisors. We begin with the residue along $\{s_2 = 1\}$.

**Proposition 5.1.** *For* $s_1, s_3, s_4, s_5 > 1$,

$$\underset{s_2=1}{\mathit{Res}}\, Z_S(s_1,s_2,s_3,s_4,s_5) = \tag{5.4}$$

$$r_S Y_S(s_1+s_3, s_4, s_5)\zeta_S(2s_1)\zeta_S(2s_3)\zeta_S(2s_3+2s_4-1)\zeta_S(2s_3+2s_4+2s_5-2),$$

*where* $r_S = \prod_{P\in S}\left(1-\frac{1}{|P|}\right)$.

*Proof.* For $s_1, s_3, s_4, s_5 > 1$,

$$\begin{aligned}
&\underset{s_2=1}{\mathrm{Res}}\, Z_S(s_1,s_2,s_3,s_4,s_5)\\
&\quad= \underset{s_2=1}{\mathrm{Res}} \sum_{\substack{I_1,\ldots,I_5\in\mathcal{I}(S)\\ I_1 I_3=\square}} \frac{\chi_{I_4}(\hat{I}_5)\chi_{I_4}(\hat{I}_3)}{\prod_{i=1}^{i=5}|I_i|^{s_i}} g(I_1,I_2,I_3,I_4,I_5)\\
&\quad= \underset{s_2=1}{\mathrm{Res}} \sum_{I_1,I_3,I_4 I_5\in\mathcal{I}(S)} \frac{\chi_{I_4}(\hat{I}_5)\chi_{I_4}(\hat{I}_3)}{|I_1|^{s_1}|I_3|^{s_3}|I_4|^{s_4}|I_5|^{s_5}} g'(I_1,I_3,I_4,I_5),
\end{aligned}$$

say, where the new coefficients $g'(I_1,I_3,I_4,I_5)$ are defined by

$$g'(I_1,I_3,I_4,I_5) = \underset{s_2=1}{\mathrm{Res}} \sum_{I_2\in\mathcal{I}(S)} \frac{g(I_1,I_2,I_3,I_4,I_5)}{|I_2|^{s_2}}. \tag{5.5}$$

The coefficients $g'$ are again multiplicative, and thus determined by the generating series

$$H'(x,z,w,v) = \sum_{j,l,m,n\geq 0} g'(P^j,P^l,P^m,P^n)x^j z^l w^m v^n$$

By the definition (5.5), we deduce that $H'(x,z,w,v)$ is

$$\begin{aligned}
&\left(1-\frac{1}{p}\right) \sum_{\substack{j,k,l,m,n\geq 0\\ j+l \text{ even}}} g(P^j,P^k,P^l,P^m,P^n)x^j y^k z^l w^m v^n\Big|_{y=\frac{1}{p}}\\
&\qquad\qquad = \frac{1}{2}\left(1-\frac{1}{p}\right)\Big[H(x,\tfrac{1}{p},z,w,v)+H(-x,\tfrac{1}{p},-z,w,v)\Big].
\end{aligned}$$

Referring to the appendix,

$$\begin{aligned}
&H'(x,z,w,v) =\\
&\quad\frac{1-vw-wxz+vwxz+pvw^2xz-pv^2w^2xz-pvw^2x^2z^2+pv^2w^3x^2z^2}{\text{denominator}}
\end{aligned}$$

where the denominator is

$$\begin{aligned}
&(1-v)(1-w)(1-pv^2w^2)(1-x^2)(1-xz)(1-z^2)\\
&\qquad(1-pw^2z^2)(1-p^2v^2w^2z^2)(1-pw^2x^2z^2)(1-p^2v^2w^2x^2z^2)
\end{aligned}$$

Comparing this with (5.1) and (5.4) completes the proof of the proposition. □

As the other polar divisors of $Z_S(s_1,s_2,s_3,s_4,s_5)$ are translates of the the divisor at $\{s_2=1\}$, by the group of functional equations, the other 7 residues can be computed by applying the appropriate functional equation to the residue in Proposition 5.1. We record these below:

- Residue at $\{s_1+s_2=3/2\}$:

$$Y_S(1-s_1+s_3,s_4,s_5)\zeta_S(2-2s_1)\zeta_S(2s_3) \zeta_S(2s_3+2s_4-1)\zeta_S(2s_3+2s_4+2s_5-2)+O(s_1-1/2) \tag{5.6}$$

- Residue at $\{s_1+s_2+s_3=2\}$:

$$Y_S(2-s_1-s_3,s_3+s_4-1/2,s_5)\zeta_S(2-2s_1)\zeta_S(2-2s_3) \zeta_S(2s_4)\zeta_S(2s_4+2s_5-1)+O(s_1+s_3-1) \tag{5.7}$$

- Residue at $\{s_2+s_3=3/2\}$:

$$Y_S(1+s_1-s_3,s_3+s_4-1/2,s_5)\zeta_S(2s_1)\zeta_S(2-2s_3) \zeta_S(2s_4)\zeta_S(2s_4+2s_5-1)+O(s_3-1/2) \tag{5.8}$$

The implicit constants in the $O$-notation are uniformly bounded in a neighborhood of $(1/2,1,1/2,1,1/2)$. The other four residues follow from the symmetry

$$Z_S(s_1,s_2,s_3,s_4,s_5)=Z_S(s_5,s_4,s_3,s_2,s_1).$$

Summing up all of these polar contributions, we find that

$$Z_S(1/2,s,1/2,s,1/2)=\frac{r_S\zeta_S(\frac{3}{2})\zeta_S(2)^3}{(s-1)^5}+O\left(\frac{1}{(s-1)^4}\right) \tag{5.9}$$

as $s\to 1$.

A closer investigation of the polar hyperplanes reveals that

$$(s-1)^5 Z_S(\tfrac{1}{2},s,\tfrac{1}{2},s,\tfrac{1}{2})$$

has an additional pole at $s=3/4$ but is analytic for $\mathrm{Re}(s)>3/4$. Also, an argument identical to that given in Proposition 4.12 of [5] shows that for any $\epsilon>0$,

$$(s-1)^5 Z_S(\tfrac{1}{2},s,\tfrac{1}{2},s,\tfrac{1}{2})$$

has polynomial growth in $\mathrm{Im}(s)$ when $\mathrm{Re}(s)>3/4+\epsilon$.

To prove Theorem 2.3 we take a smooth, non-negative test function $f$ compactly supported in $(0,\infty)$. Thus the Mellin transform $\tilde{f}$ has superpolynomial decay in vertical strips of bounded width. By the Mellin inversion formula, we obtain

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z_S(\tfrac{1}{2},s,\tfrac{1}{2},s,\tfrac{1}{2})\tilde{f}(s)X^s ds = \sum_{I_2,I_4\in\mathcal{I}(S)} a(I_2,I_4)L_S(\tfrac{1}{2},\chi_{I_2})L_S(\tfrac{1}{2},\chi_{I_4})L_S(\tfrac{1}{2},\chi_{I_2I_4})f\left(\tfrac{|I_2I_4|}{X}\right). \tag{5.10}$$

On the other hand, moving the line of integration to $\mathrm{Re}(s) = 3/4 + \epsilon$, we pick up the pole of order 5 at $s = 1$, which has a residue of the form

$$A_4 X(\log X)^4 + A_3 X(\log X)^3 + A_2 X(\log X)^2 + A_1 X(\log X) + A_0 X,$$

where

$$A_4 = \tilde{f}(1)\frac{\zeta_S(\frac{3}{2})\zeta_S(2)^3}{4!}\prod_{P\in S}\left(1 - \frac{1}{|P|}\right)$$

and the other constants $A_0, \ldots, A_3$ can also be computed explicitly in terms of derivatives of $\tilde{f}$ at 1. The integral at $\mathrm{Re}(s) = 3/4+\epsilon$ converges absolutely and provides the error term $O(X^{3/4+\epsilon})$, with the implicit constant depending on $\epsilon, K$ and $S$.

## References

[1] B. Brubaker, D. Bump, in preparation.
[2] B. Brubaker, D. Bump, G. Chinta, S. Friedberg and J. Hoffstein, in preparation.
[3] D. Bump, S. Friedberg and J. Hoffstein, On some applications of automorphic forms to number theory. *Bull. A.M.S.* **33** (1996), 157–175.
[4] D. Bump, S. Friedberg and J. Hoffstein, Sums of twisted GL(3) automorphic $L$-functions. *To appear in a volume dedicated to Joseph Shalika*.
[5] A. Diaconu, D. Goldfeld and J. Hoffstein, Multiple Dirichlet series and moments of zeta and $L$-functions. Compositio Math. 139 (2003), no. 3,297–360.
[6] B. Fisher and S. Friedberg, Double Dirichlet series over function fields. Compos. Math. 140 (2004), no. 3, 613–630.
[7] S. Friedberg and J. Hoffstein, Nonvanishing theorems for automorphic $L$-functions on GL(2). *Ann. of Math. (2)*, **142**, (1995), 385–423.
[8] C. F. Gauss, Disquisitiones arithmeticae. Translated into English by Arthur A. Clarke, S. J. Yale University Press, New Haven, Conn.-London 1966 xx+472 pp.
[9] D. Goldfeld and J. Hoffstein, Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet $L$-series. Invent. Math. 80 (1985), no. 2, 185–208.
[10] D. Goldfeld and C. Viola, Mean values of $L$-functions associated to elliptic, Fermat and other curves at the centre of the critical strip. J. Number Theory 11 (1979), no. 3 S. Chowla Anniversary Issue, 305–320.
[11] M. Jutila, On the mean value of $L(\frac{1}{2}, \chi)$ for real characters. Analysis 1 (1981), no. 2, 149–161.
[12] R. Lipschitz, Sitzungsber. Akad. Berlin (1865), pp. 174-185.
[13] C. L. Siegel, The average measure of quadratic forms with given determinant and signature. Ann. of Math. (2) 45, (1944). 667–685.
[14] C. L. Siegel, Die Funktionalgleichungen einiger Dirichletscher Reihen. (German) Math. Z. 63 (1956), 363–373.
[15] L. A. Takhtadzjan and A. I. Vinogradov, On analogues of the Gauss-Vinogradov formula. (Russian) Dokl. Akad. Nauk SSSR 254 (1980), no. 6, 1298–1301. {English translation: Soviet Math. Dokl. 22 (1980), no. 2, 555–559.}

## 6. Appendix

In this Appendix we present the rational function whose properties were described in Proposition 4.1. The reader wishing to check the proposition may download the rational function from

$$http://www.math.brown.edu/\tilde{\ }chinta/a5poly$$

We have

$$H(x,y,z,w,v)=\frac{\text{numerator}}{\text{denominator}},$$

where the denominator is

$$(1-x)\,(1-y)\,(1-z)\,(1-w)\,(1-v)\,\left(1-p\,v^2\,w^2\right)\,\left(1-p\,x^2\,y^2\right) \left(1-p\,w^2\,z^2\right)\,\left(1-p^2\,v^2\,w^2\,z^2\right)\,\left(1-p\,y^2\,z^2\right)\,\left(1-p^2\,w^2\,y^2\,z^2\right) \left(1-p^3\,v^2\,w^2\,y^2\,z^2\right)\,\left(1-p^2\,x^2\,y^2\,z^2\right)\,\left(1-p^3\,w^2\,x^2\,y^2\,z^2\right)\,\left(1-p^4\,v^2\,w^2\,x^2\,y^2\,z^2\right) \tag{6.1}$$

and the numerator is given by

$$\begin{aligned}&1-vw-xy+vwxy-wz+vwz+\\
&pv^2w^2z-yz+wyz-pvw^2yz+pv^2w^2yz+xyz-vwxyz+pxy^2z-\\
&pwxy^2z+p^2vw^2xy^2z-p^2v^2w^2xy^2z-px^2y^2z+pwx^2y^2z-p^2vw^2x^2y^2z+\\
&p^2v^2w^2x^2y^2z-pvw^2z^2+pv^2w^3z^2+pwyz^2-pvwyz^2-pw^2yz^2+pvw^2yz^2+\\
&pvw^3yz^2-pv^2w^3yz^2-p^2v^2w^3yz^2+p^2v^3w^3yz^2-pwxyz^2+pvwxyz^2+pw^2xyz^2-\\
&pvw^3xyz^2+p^2v^2w^3xyz^2-p^2v^3w^3xyz^2-pwy^2z^2+pvwy^2z^2-p^2v^2w^2y^2z^2+\\
&p^2v^2w^3y^2z^2-pxy^2z^2+pwxy^2z^2-p^2vw^2xy^2z^2+p^2v^2w^2xy^2z^2-p^2w^2x^2y^2z^2+\\
&p^2vw^2x^2y^2z^2-p^3v^2w^2x^2y^2z^2+p^2vw^3x^2y^2z^2-p^2v^2w^3x^2y^2z^2+\\
&p^3v^3w^3x^2y^2z^2+pwxy^3z^2-pvwxy^3z^2+p^2v^2w^2xy^3z^2-p^2v^2w^3xy^3z^2+px^2y^3z^2\\
&-pwx^2y^3z^2-p^2wx^2y^3z^2+p^2vwx^2y^3z^2+p^2w^2x^2y^3z^2-p^2v^2w^2x^2y^3z^2\\
&-p^2vw^3x^2y^3z^2+p^2v^2w^3x^2y^3z^2+p^3v^2w^3x^2y^3z^2-p^3v^3w^3x^2y^3z^2\\
&+p^2wx^3y^3z^2-p^2vwx^3y^3z^2+p^3v^2w^2x^3y^3z^2-p^3v^2w^3x^3y^3z^2+pvw^2yz^3\\
&-pvw^3yz^3+p^2v^2w^4yz^3-p^2v^3w^4yz^3-pvw^2xyz^3+pvw^3xyz^3-p^2v^2w^4xyz^3\\
&+p^2v^3w^4xyz^3+pw^2y^2z^3-pvw^2y^2z^3-p^2vw^2y^2z^3+p^2v^2w^2y^2z^3+\\
&p^2v^2w^3y^2z^3-p^2v^3w^3y^2z^3-p^2v^2w^4y^2z^3+p^2v^3w^4y^2z^3+pwxy^2z^3-\\
&pvwxy^2z^3-pw^2xy^2z^3-p^2w^2xy^2z^3+pvw^2xy^2z^3+p^3vw^2xy^2z^3-\\
&p^3v^2w^2xy^2z^3+p^2w^3xy^2z^3-p^2v^2w^3xy^2z^3+p^2v^3w^3xy^2z^3-p^3vw^4xy^2z^3+\\
&p^2v^2w^4xy^2z^3+p^3v^2w^4xy^2z^3-p^2v^3w^4xy^2z^3-p^4v^3w^4xy^2z^3+p^4v^4w^4xy^2z^3+\\
&p^2w^2x^2y^2z^3-p^3vw^2x^2y^2z^3+p^3v^2w^2x^2y^2z^3-p^2vw^3x^2y^2z^3+p^3v^2w^3x^2y^2z^3-\end{aligned} \tag{6.2}$$

$$p^3v^3w^3x^2y^2z^3 + p^2vw^2y^3z^3 - p^2v^2w^3y^3z^3 - pwxy^3z^3 + pvwxy^3z^3 -$$
$$p^2v^2w^2xy^3z^3 + p^2v^2w^3xy^3z^3 + p^2w^2x^2y^3z^3 - p^2vw^2x^2y^3z^3 + p^3v^2w^2x^2y^3z^3 -$$
$$p^2w^3x^2y^3z^3 + p^2vw^3x^2y^3z^3 - p^3v^2w^3x^2y^3z^3 + p^3vw^4x^2y^3z^3 - 2p^3v^2w^4x^2y^3z^3 +$$
$$p^3v^3w^4x^2y^3z^3 + p^4v^3w^4x^2y^3z^3 - p^4v^4w^4x^2y^3z^3 - p^2w^2x^3y^3z^3 + p^2vw^2x^3y^3z^3 -$$
$$p^3v^2w^2x^3y^3z^3 + p^3v^3w^3x^3y^3z^3 + p^3v^2w^4x^3y^3z^3 - p^3v^3w^4x^3y^3z^3 - p^3vw^2xy^4z^3 +$$
$$p^3v^2w^3xy^4z^3 + p^2wx^2y^4z^3 - p^2vwx^2y^4z^3 - p^2w^2x^2y^4z^3 + p^2vw^2x^2y^4z^3 +$$
$$p^3vw^2x^2y^4z^3 - 2p^3v^2w^3x^2y^4z^3 + p^3v^3w^3x^2y^4z^3 + p^3v^2w^4x^2y^4z^3 - p^3v^3w^4x^2y^4z^3 -$$
$$p^2wx^3y^4z^3 + p^2vwx^3y^4z^3 + p^2w^2x^3y^4z^3 - p^2vw^2x^3y^4z^3 - p^4vw^2x^3y^4z^3 +$$
$$p^3v^2w^3x^3y^4z^3 + p^4v^2w^3x^3y^4z^3 - p^3v^3w^3x^3y^4z^3 - p^3v^2w^4x^3y^4z^3 + p^3v^3w^4x^3y^4z^3 +$$
$$p^4vw^2x^4y^4z^3 - p^4v^2w^3x^4y^4z^3 + p^2vw^3y^2z^4 - p^2v^2w^3y^2z^4 + p^3v^3w^4y^2z^4 -$$
$$p^3v^3w^5y^2z^4 + p^2vw^2xy^2z^4 - p^2vw^3xy^2z^4 - p^3vw^3xy^2z^4 + p^3v^2w^3xy^2z^4 +$$
$$p^3vw^4xy^2z^4 - p^3v^3w^4xy^2z^4 - p^3v^2w^5xy^2z^4 + p^3v^3w^5xy^2z^4 + p^4v^3w^5xy^2z^4 -$$
$$p^4v^4w^5xy^2z^4 + p^3vw^3x^2y^2z^4 - p^3v^2w^3x^2y^2z^4 + p^4v^3w^4x^2y^2z^4 - p^4v^3w^5x^2y^2z^4 -$$
$$p^2vw^3y^3z^4 + p^2v^2w^3y^3z^4 + p^3v^2w^4y^3z^4 - p^3v^3w^4y^3z^4 + p^2w^2xy^3z^4 -$$
$$p^2vw^2xy^3z^4 - p^3vw^2xy^3z^4 + p^3v^2w^2xy^3z^4 - p^2w^3xy^3z^4 + p^2vw^3xy^3z^4 +$$
$$p^3vw^3xy^3z^4 - p^3v^2w^3xy^3z^4 - 2p^3v^2w^4xy^3z^4 + p^3v^3w^4xy^3z^4 + p^4v^3w^4xy^3z^4 -$$
$$p^4v^4w^4xy^3z^4 + p^3v^2w^5xy^3z^4 - p^4v^3w^5xy^3z^4 + p^4v^4w^5xy^3z^4 - p^2w^2x^2y^3z^4 +$$
$$p^3vw^2x^2y^3z^4 - p^3v^2w^2x^2y^3z^4 + p^2w^3x^2y^3z^4 - p^3vw^3x^2y^3z^4 + p^3v^2w^3x^2y^3z^4 -$$
$$p^3vw^4x^2y^3z^4 + p^3v^2w^4x^2y^3z^4 + p^4v^2w^4x^2y^3z^4 - 2p^4v^3w^4x^2y^3z^4 + p^4v^4w^4x^2y^3z^4 -$$
$$p^4v^2w^4x^3y^3z^4 + p^4v^3w^5x^3y^3z^4 + p^3vw^2xy^4z^4 - p^3v^2w^3xy^4z^4 + p^3w^3x^2y^4z^4 -$$
$$2p^3vw^3x^2y^4z^4 + p^3v^2w^3x^2y^4z^4 + p^4v^2w^3x^2y^4z^4 - p^4v^3w^3x^2y^4z^4 +$$
$$p^4v^2w^4x^2y^4z^4 - p^4v^3w^4x^2y^4z^4 + p^5v^4w^4x^2y^4z^4 - p^4v^2w^5x^2y^4z^4 + p^4v^3w^5x^2y^4z^4 -$$
$$p^5v^4w^5x^2y^4z^4 + p^3w^2x^3y^4z^4 - p^3vw^2x^3y^4z^4 + p^4v^2w^2x^3y^4z^4 - p^3w^3x^3y^4z^4 +$$
$$p^3vw^3x^3y^4z^4 + p^4vw^3x^3y^4z^4 - 2p^4v^2w^3x^3y^4z^4 - p^4v^2w^4x^3y^4z^4 + p^4v^3w^4x^3y^4z^4 +$$
$$p^5v^3w^4x^3y^4z^4 - p^5v^4w^4x^3y^4z^4 + p^4v^2w^5x^3y^4z^4 - p^4v^3w^5x^3y^4z^4 - p^5v^3w^5x^3y^4z^4 +$$
$$p^5v^4w^5x^3y^4z^4 - p^4vw^3x^4y^4z^4 + p^4v^2w^3x^4y^4z^4 + p^5v^2w^4x^4y^4z^4 - p^5v^3w^4x^4y^4z^4 -$$
$$p^3vw^2x^2y^5z^4 + p^3vw^3x^2y^5z^4 - p^4v^2w^4x^2y^5z^4 + p^4v^3w^4x^2y^5z^4 - p^3w^2x^3y^5z^4 +$$
$$p^3vw^2x^3y^5z^4 + p^4vw^2x^3y^5z^4 - p^4v^2w^2x^3y^5z^4 - p^4vw^3x^3y^5z^4 + p^4v^3w^3x^3y^5z^4 +$$
$$p^4v^2w^4x^3y^5z^4 - p^4v^3w^4x^3y^5z^4 - p^5v^3w^4x^3y^5z^4 + p^5v^3w^5x^3y^5z^4 - p^4vw^2x^4y^5z^4 +$$
$$p^4vw^3x^4y^5z^4 - p^5v^2w^4x^4y^5z^4 + p^5v^3w^4x^4y^5z^4 - p^3v^2w^4y^3z^5 + p^3v^3w^5y^3z^5 +$$
$$p^4vw^3xy^3z^5 - p^4v^2w^3xy^3z^5 - p^4vw^4xy^3z^5 + p^3v^2w^4xy^3z^5 + p^4v^2w^4xy^3z^5 -$$

$$p^3v^3w^5xy^3z^5-p^5v^3w^5xy^3z^5+p^5v^4w^5xy^3z^5+p^5v^3w^6xy^3z^5-p^5v^4w^6xy^3z^5-$$
$$p^4vw^3x^2y^3z^5+p^4v^2w^3x^2y^3z^5+p^4vw^4x^2y^3z^5-2p^4v^2w^4x^2y^3z^5+p^4v^3w^5x^2y^3z^5+$$
$$p^5v^3w^5x^2y^3z^5-p^5v^4w^5x^2y^3z^5-p^5v^3w^6x^2y^3z^5+p^5v^4w^6x^2y^3z^5+p^4v^2w^4x^3y^3z^5-$$
$$p^4v^3w^5x^3y^3z^5-p^4vw^3xy^4z^5+p^4v^2w^3xy^4z^5+p^4vw^4xy^4z^5-p^4v^2w^5xy^4z^5+$$
$$p^5v^3w^5xy^4z^5-p^5v^4w^5xy^4z^5-p^3w^3x^2y^4z^5+p^3vw^3x^2y^4z^5+p^4vw^3x^2y^4z^5-$$
$$2p^4v^2w^3x^2y^4z^5+p^4v^3w^3x^2y^4z^5-p^4v^2w^4x^2y^4z^5+p^5v^3w^4x^2y^4z^5-p^5v^4w^4x^2y^4z^5+$$
$$p^4v^2w^5x^2y^4z^5-p^5v^3w^5x^2y^4z^5+p^5v^4w^5x^2y^4z^5+p^5v^2w^4x^3y^4z^5-p^5v^2w^5x^3y^4z^5+$$
$$p^6v^3w^6x^3y^4z^5-p^6v^4w^6x^3y^4z^5-p^5v^2w^4x^4y^4z^5+p^5v^3w^5x^4y^4z^5-p^4vw^4x^2y^5z^5+$$
$$p^4v^2w^4x^2y^5z^5-p^5v^3w^4x^2y^5z^5+p^4v^2w^5x^2y^5z^5-p^4v^3w^5x^2y^5z^5+$$
$$p^5v^4w^5x^2y^5z^5+p^3w^3x^3y^5z^5-p^3vw^3x^3y^5z^5-p^5vw^3x^3y^5z^5+p^4v^2w^3x^3y^5z^5+$$
$$p^5v^2w^3x^3y^5z^5-p^4v^3w^3x^3y^5z^5+p^5vw^4x^3y^5z^5-p^5v^2w^4x^3y^5z^5+$$
$$p^5v^4w^4x^3y^5z^5-p^4v^2w^5x^3y^5z^5+p^4v^3w^5x^3y^5z^5+p^6v^3w^5x^3y^5z^5-$$
$$p^5v^4w^5x^3y^5z^5-p^6v^4w^5x^3y^5z^5-p^6v^3w^6x^3y^5z^5+p^6v^4w^6x^3y^5z^5+$$
$$p^5vw^3x^4y^5z^5-p^5v^2w^3x^4y^5z^5-p^5vw^4x^4y^5z^5+p^5v^2w^4x^4y^5z^5+$$
$$p^5v^2w^5x^4y^5z^5-p^5v^3w^5x^4y^5z^5-p^6v^3w^5x^4y^5z^5+p^6v^4w^5x^4y^5z^5+p^5vw^3x^3y^6z^5-$$
$$p^5v^2w^3x^3y^6z^5+p^6v^3w^4x^3y^6z^5-p^6v^3w^5x^3y^6z^5-p^5vw^3x^4y^6z^5+p^5v^2w^3x^4y^6z^5-$$
$$p^6v^3w^4x^4y^6z^5+p^6v^3w^5x^4y^6z^5-p^4v^2w^4xy^4z^6+p^4v^2w^5xy^4z^6-p^5v^3w^6xy^4z^6+$$
$$p^5v^4w^6xy^4z^6-p^4vw^4x^2y^4z^6+p^4v^2w^4x^2y^4z^6+p^5v^2w^4x^2y^4z^6-p^5v^3w^4x^2y^4z^6-$$
$$p^5v^2w^5x^2y^4z^6+p^5v^4w^5x^2y^4z^6+p^5v^3w^6x^2y^4z^6-p^5v^4w^6x^2y^4z^6-p^6v^4w^6x^2y^4z^6+$$
$$p^6v^4w^7x^2y^4z^6-p^5v^2w^4x^3y^4z^6+p^5v^2w^5x^3y^4z^6-p^6v^3w^6x^3y^4z^6+p^6v^4w^6x^3y^4z^6+$$
$$p^4vw^4x^2y^5z^6-p^5v^2w^4x^2y^5z^6+p^5v^3w^4x^2y^5z^6-p^4v^2w^5x^2y^5z^6+p^5v^3w^5x^2y^5z^6-$$
$$p^5v^4w^5x^2y^5z^6+p^5v^2w^5x^3y^5z^6-p^5v^3w^5x^3y^5z^6+p^6v^4w^6x^3y^5z^6-p^6v^4w^7x^3y^5z^6+$$
$$p^5v^2w^4x^4y^5z^6-p^5v^2w^5x^4y^5z^6+p^6v^3w^6x^4y^5z^6-p^6v^4w^6x^4y^5z^6-p^5vw^4x^3y^6z^6+$$
$$p^5v^2w^4x^3y^6z^6-p^6v^3w^4x^3y^6z^6+p^6v^4w^5x^3y^6z^6+p^6v^3w^6x^3y^6z^6-p^6v^4w^6x^3y^6z^6+$$
$$p^5vw^4x^4y^6z^6-p^5v^2w^4x^4y^6z^6-p^6v^2w^4x^4y^6z^6+p^6v^3w^4x^4y^6z^6+p^6v^3w^5x^4y^6z^6-$$
$$p^6v^4w^5x^4y^6z^6-p^6v^3w^6x^4y^6z^6+p^6v^4w^6x^4y^6z^6+p^6v^2w^4x^4y^7z^6-p^6v^3w^5x^4y^7z^6+$$
$$p^5v^2w^5x^2y^5z^7-p^5v^3w^5x^2y^5z^7+p^6v^4w^6x^2y^5z^7-p^6v^4w^7x^2y^5z^7-p^5v^2w^5x^3y^5z^7+$$
$$p^5v^3w^5x^3y^5z^7-p^6v^4w^6x^3y^5z^7+p^6v^4w^7x^3y^5z^7-p^7v^3w^6x^3y^6z^7+p^7v^4w^7x^3y^6z^7+$$
$$p^6v^2w^5x^4y^6z^7-p^6v^3w^5x^4y^6z^7+p^7v^4w^6x^4y^6z^7-p^7v^4w^7x^4y^6z^7-p^6v^2w^5x^4y^7z^7+$$
$$p^6v^3w^5x^4y^7z^7+p^7v^3w^6x^4y^7z^7-p^7v^4w^6x^4y^7z^7+p^7v^3w^6x^3y^6z^8-p^7v^4w^7x^3y^6z^8-$$
$$p^7v^3w^6x^4y^7z^8+p^7v^4w^7x^4y^7z^8$$