NON-VANISHING TWISTS OF GL(2) AUTOMORPHIC L-FUNCTIONS

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Abstract. Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$. Suppose there exists a single non-vanishing n^{th} order twist of the L-series associated to π at the center of the critical strip. We use the method of multiple Dirichlet series to establish that there exist infinitely many such non-vanishing n^{th} order twists of the L-series of the representation at the center.

1. Introduction and Statement of the Main Result

Let E be an elliptic curve defined over a number field K. The behavior of the rank of the L-rational points $E(L)$ as L varies over some family of algebraic extensions of K is a problem of fundamental interest. The conjecture of Birch and Swinnerton-Dyer provides a means to investigate this problem via the theory of automorphic L-functions.

Assume that the L-function of E coincides with the L-function $L(s, \pi)$ of a cuspidal automorphic representation of GL(2) of the adele ring $\mathbb{A}_K$. Let L/K be a finite cyclic extension and χ a Galois character of this extension. Then the conjecture of Birch and Swinnerton-Dyer equates the rank of the χ-isotypic component $E(L)^\chi$ of $E(L)$ with the order of vanishing of the twisted L-function $L(s, \pi \otimes \chi)$ at the central point $s = \frac{1}{2}$. In particular, the χ-component $E(L)^\chi$ is finite (according to the conjecture) if and only if the central value $L(\frac{1}{2}, \pi \otimes \chi)$ is non-zero.

Thus it becomes of arithmetic interest to establish non-vanishing results for central values of twists of automorphic L-functions by characters of finite order. For quadratic twists this problem has received much attention in recent years. In this paper we address this question for twists of higher order. Our main result is contained in the following theorem.

Theorem 1.1. *Fix a prime integer $n > 2$, a number field K containing the n^{th} roots of unity, and a sufficiently large finite set of primes S of K. Let π be a self-contragredient cuspidal automorphic representation*

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of $\mathrm{GL}(2,\mathbb{A}_K)$ *which has trivial central character and is unramified outside* S*. Suppose there exists an idèle class character* χ_0 *of* K *of order* n *unramified outside* S *such that*

$$L(\tfrac{1}{2},\pi\otimes\chi_0)\neq 0.$$

Then there exist infinitely many idèle class characters χ *of* K *of order* n *unramified outside* S *such that*

$$L(\tfrac{1}{2},\pi\otimes\chi)\neq 0.$$

We refer the reader to Section 2 for the definition of the finite set S. Fearnley and Kisilevsky have proven a related result for the L-function $L(s,E)$ of an elliptic curve defined over $\mathbb{Q}$. In [5] they show that if the algebraic part $L^{\mathrm{alg}}(\frac{1}{2},E)$ of the central L-value is nonzero mod n, then there exist infinitely many Dirichlet characters χ of order n such that $L(\frac{1}{2},E,\chi)\neq 0$. We note that if $L(\frac{1}{2},E)\neq 0$ then the hypothesis $L^{\mathrm{alg}}(\frac{1}{2},E)\not\equiv 0$ mod n is satisfied for all sufficiently large primes n. We find it interesting (and frustrating!) that, although the methods of [5] (based on the arithmetic of modular symbols) are completely different from the methods of this paper, both our result and theirs require some nonvanishing assumption. An unconditional result in the cubic case ($n=3$) has recently been established in [1]. We comment more on this below.

The quadratic case ($n=2$) is particularly accessible because, by the results of Waldspurger [17], Kohnen and Zagier [13], and others, the existence of a quadratic character χ such that $L(\frac{1}{2},\pi\otimes\chi)\neq 0$ implies the existence of a metaplectic cuspidal automorphic representation $\widetilde{\pi}$ on the double cover of $\mathrm{GL}(2,\mathbb{A}_K)$ corresponding to π. The correspondent $\widetilde{\pi}$ is related to π in the following way. If $L(w,\widetilde{\pi}\otimes\widetilde{\pi})$ denotes the Rankin-Selberg convolution of $\widetilde{\pi}$ with itself then

$$L(w,\widetilde{\pi}\otimes\widetilde{\pi})=\sum_{d\neq 0}\frac{L(\frac{1}{2},\pi\otimes\chi_{d_0}^{(2)})P_d(\pi)}{\mathbb{N}d^w}, \tag{1.1}$$

up to corrections at a finite number of places. Here $d=d_0d_2^2$, with d_0 square free, the $\chi_{d_0}^{(2)}$ are quadratic characters with conductor d_0, and the $P_d(\pi)$ are certain non-zero *correction factors* which are trivial when $d_2=1$. We defer a precise definition of such objects until Section 2. These correction factors are small in the sense that, for any fixed d_0,

$$\text{(1.2)}\qquad \textit{the sum } \sum_{d_2\neq 0}\frac{P_{d_0d_2^2}(\pi)}{\mathbb{N}d_2^{2w}} \textit{ converges absolutely for any } w \textit{ with } \mathrm{Re}(w)>\tfrac{1}{2}.$$

This connection between π and $\widetilde{\pi}$ causes the existence of one non-vanishing quadratic twist to imply the existence of infinitely many d_0 such that $L(\frac{1}{2}, \pi \otimes \chi_{d_0}^{(2)}) \neq 0$. This is because if $\widetilde{\pi} \neq 0$ then $L(w, \widetilde{\pi} \otimes \widetilde{\pi})$ has a pole at $w = 1$. However the right hand side of (1.1) will converge at $w = 1$ if there are only finitely many non-vanishing quadratic twists.

For $n > 2$ there are no known results relating n^{th} order twists of the L-series of π to Fourier coefficients of other automorphic objects. In fact even a conjectural generalization of the results of Waldspurger to the case $n > 2$ remains mysterious. However, in this paper we describe how a generalization of (1.1) can still be found by associating π to a certain metaplectic form. This generalization is at least sufficient to answer the question of whether one non-vanishing twist of a given order implies the existence of infinitely many non-vanishing twists of that order. It may ultimately shed some light on the question of the correct generalization of Waldspurger's results, but at the moment this aspect remains opaque.

We will describe in detail a Dirichlet series that has the rough form

$$Z^{(n)}(s,w) = \sum_{d \neq 0} \frac{L(s, \pi \otimes \overline{\chi}_{d_0}^{(n)}) \epsilon(d_0) P_d(s,\pi)}{\mathbb{N}d^w}, \tag{1.3}$$

where $d = d_0 d_n^n$ with d_0 n^{th} power-free (see Section 2). Here $\epsilon(d_0)$ denotes an n^{th} order Gauss sum corresponding to the character $\chi_{d_0}^{(n)}$ and $P_d(s,\pi)$ again denotes certain correction factors which are trivial when $d_n = 1$. These are also small, in the sense that for $\mathrm{Re}(s) \geq \frac{1}{2}$,

$$\textit{the sum } \sum_{d_n \neq 0} \frac{P_{d_0 d_n^n}(s,\pi)}{\mathbb{N}d_n^{nw}} \textit{ converges absolutely for any } w \textit{ with } \mathrm{Re}(w) > \frac{1}{n} + \frac{1}{9}. \tag{1.4}$$

The fraction $\frac{1}{9}$ comes from the bound of Kim and Shahidi [12].

The series $Z^{(n)}(s,w)$ is "natural" for the following reasons. First, when $n = 2$ and $\widetilde{\pi}$ exists, $Z^{(n)}(\frac{1}{2}, w)$ agrees at almost all places with the Rankin-Selberg convolution $L(w, \widetilde{\pi} \otimes \widetilde{\pi})$. Second, after an interchange in the order of summation, it has an automorphic interpretation as a Rankin-Selberg convolution of π with an Eisenstein series on the n-fold cover of GL(2). In the case $n = 2$, this automorphic interpretation of $Z^{(n)}(s,w)$ was exploited by Friedberg and Hoffstein [8]. In the case $n = 3$ the automorphic interpretation was used by She [16] to establish a non-vanishing result for cubic twists of one particular π.

In this paper we do not use the automorphic interpretation of $Z^{(n)}(s,w)$. Instead we take the far easier approach of the method of multiple Dirichlet series (discussed in brief at the conclusion of this section).

Using this method we establish an analytic continuation and exhibit a finite group of functional equations for $Z^{(n)}(s,w)$ in the two variables s and w. Specializing to $s=\frac{1}{2}$, we obtain a Dirichlet series $Z^{(n)}(\frac{1}{2},w)$ with a functional equation in w. The condition (1.4) implies that if $L(\frac{1}{2},\pi\otimes\overline{\chi}^{(n)}_{d_0})\neq 0$ for only finitely many d_0 then $Z^{(n)}(\frac{1}{2},w)$ must converge for $\mathrm{Re}(w)>\frac{1}{n}+\frac{1}{9}$. We then show that for $n\geq 3$ (i.e. $\frac{1}{n}+\frac{1}{9}<\frac{1}{2}$), this is incompatible with the functional equation. It immediately follows that the existence of one non-vanishing twist implies the existence of infinitely many.

The method can easily be taken a bit further to establish a mean value result of the form, for $\mathrm{Re}(s)>\frac{1}{2}$,

$$\sum L(s,\pi\otimes\overline{\chi}^{(n)}_{d_0})\,\epsilon(d_0)\,P_d(\tfrac{1}{2},\pi)\,W\left(\frac{\mathbb{N}d}{X}\right)\sim c(s,\pi)X^{\frac{1}{2}+\frac{1}{n}}, \tag{1.5}$$

where W is any suitable smoothing function. The constant $c(s,\pi)$ is a very interesting function; it is a simple multiple of $L(s+\frac{1}{2n},\pi\otimes\theta^{(n)})$, the Rankin-Selberg convolution of π with the theta function on the n-fold cover of GL(2), evaluated at the point $s+\frac{1}{2n}$. When $n=2$ and $s=\frac{1}{2}$, the series $L(\frac{3}{4},\pi\otimes\theta^{(2)})$ is essentially the symmetric square L-series of π evaluated at 1, the edge of the critical strip. Thus in this case the L-series does not vanish, and simple conditions on the sign of the functional equation of π determine whether or not $c(\frac{1}{2},\pi)$ equals zero. Because of this, unconditional mean value and non-vanishing results can be derived, as was done in [8]. For $n>2$, however, the L-series $L(\frac{1}{2}+\frac{1}{2n},\pi\otimes\theta^{(n)})$ does not have an Euler product and is evaluated at a point inside the critical strip. Thus the question of vanishing becomes quite subtle. It is because of this that we cannot yet eliminate the possibility that the entire class of twists $L(\frac{1}{2},\pi\otimes\overline{\chi}^{(n)}_{d_0})$ vanishes identically. In [1] a different multiple Dirichlet series is constructed, specific to the case $n=3$. In this case the (cubic) Gauss sum is removed from the numerator and the constant $c(s,\pi)$ becomes essentially $L(3s,\pi,\mathrm{sym}^3)$. As a consequence unconditional non-vanishing and mean value results can be obtained in the case $n=3$. The question of generalizing this method to $n\geq 4$ remains open and extremely interesting.

We close the introduction with a brief overview of the method of multiple Dirichlet series. Multiple Dirichlet series are functions of several complex variables of the form

$$\sum_{m_1,\ldots,m_r}\frac{a(m_1,m_2,\ldots,m_r)}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}}.$$

These can be considered, according to the order of summation, as a Dirichlet series in any one of the variables whose coefficients are again Dirichlet series. For example, in (1.3) the multiple Dirichlet series $Z^{(n)}(s,w)$ is a Dirichlet series in the variable w with numerator $L(s,\pi\otimes\overline{\chi}^{(n)}_{d_0})\epsilon(d_0)P_d(s,\pi)$, a family of Dirichlet series in the variable s. If a component Dirichlet series possesses a functional equation, then the multiple Dirichlet series inherits a corresponding functional equation. Interchanges in the order of summation may reveal new families of Dirichlet series in the numerator with new functional equations. Interchanging the order of summation in (1.3) produces a Dirichlet series formed from n^{th} order Gauss sums. Such series arise in the theory of Eisenstein series on the n-fold cover of GL_2 as introduced by Kubota in [14], and extended by Patterson [15] and Kazhdan-Patterson [11].

Classical convexity estimates on the constituent Dirichlet series give a region of absolute convergence for the multiple Dirichlet series. Once exact functional equations are obtained, one can apply them to the domain of convergence to obtain a new domain which has a non-empty intersection with the original. This provides the analytic continuation to the union of the original domain and its translates. An analytic continuation to the convex hull of this union follows from a convexity theorem for several complex variables. In the case of $Z^{(n)}(s,w)$, we will show that we obtain a region whose convex hull is all of the complex space $\mathbb{C}^2$.

This approach was first detailed by Bump, Friedberg, and Hoffstein in [2] and [3] where instances of multiple Dirichlet series possessing these properties were catalogued. Fisher and Friedberg [6] generalized these arguments to quadratic twists of automorphic forms on GL(2) over arbitrary function fields. This method was also carried out by Friedberg, Hoffstein and Lieman [9] on a multiple Dirichlet series whose coefficients were weighted n^{th} order Dirichlet L-series in order to determine mean-value estimates for these L-series. Finally, see [1] for a different and considerably more complicated construction in the case $n = 3$ and GL(2).

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2. Preliminaries and Outline of Method

Fix $n > 2$ and let K be a number field containing the n^{th} roots of unity. Let $\mathcal{O}$ denote the ring of integers of K. Let π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_K)$. Let S_f be a finite set of non-archimedean places such that S_f contains all places dividing n, the ring of S_f-integers $\mathcal{O}_{S_f}$ has class number 1, and π is unramified outside S_f. Let S_∞ denote the set of archimedean places and let $S = S_f \cup S_\infty$.

Let $\left(\frac{a}{*}\right)$ be the power residue symbol attached to the extension $K(\sqrt[n]{a})$ of K. We extend the n^{th} power residue symbol as in Fisher and Friedberg [6]. We review the definition.

For each place v, let K_v denote the completion of K at v. For v nonarchimedean, let P_v denote the corresponding ideal of $\mathcal{O}$, and let $q_v = \mathbb{N}P_v$ denote its norm. Let $C = \prod_{v\in S_f} P_v^{n_v}$ with $n_v \geq 1$ sufficiently large so that if $a \in K_v$, and $\mathrm{ord}_v(a-1) \geq n_v$, then $a \in (K_v^\times)^n$. Let H_C be the ray class group modulo C and let $R_C = H_C \otimes \mathbb{Z}/n\mathbb{Z}$. Write the finite group R_C as a direct product of cyclic groups, choose a generator for each, and let $\mathcal{E}_0$ be a set of ideals of $\mathcal{O}$ prime to S which represent these generators. For each $E_0 \in \mathcal{E}_0$ choose $m_{E_0} \in K^\times$ such that $E_0\mathcal{O}_{S_f} = m_{E_0}\mathcal{O}_{S_f}$. Let $\mathcal{E}$ be a full set of representatives for R_C of the form $\prod_{E_0\in\mathcal{E}_0} E_0^{n_{E_0}}$, with $n_{E_0} \in \mathbb{Z}$. If $E = \prod_{E_0\in\mathcal{E}_0} E_0^{n_{E_0}}$ is such a representative, then let $m_E = \prod_{E_0\in\mathcal{E}_0} m_{E_0}^{n_{E_0}}$. Note that $E\mathcal{O}_{S_f} = m_E\mathcal{O}_{S_f}$ for all $E \in \mathcal{E}$. For convenience we suppose that $\mathcal{O} \in \mathcal{E}$ and $m_{\mathcal{O}} = 1$.

Let $\mathcal{J}(S)$ denote the group of fractional ideals of $\mathcal{O}$ coprime to S_f. Let $I, J \in \mathcal{J}(S)$ be coprime. Write $I = (m)EG^n$ with $E \in \mathcal{E}$, $m \in K^\times$, $m \equiv 1 \bmod C$, and $G \in \mathcal{J}(S)$ such that $(G, J) = 1$. Then as in [6], the n^{th} power residue symbol $\left(\frac{mm_E}{J}\right)$ is defined, and if $I = (m')E'G'^n$ is another such decomposition, then $E' = E$ and $\left(\frac{m'm_E}{J}\right) = \left(\frac{mm_E}{J}\right)$.

In view of this define the n^{th} power residue symbol $\left(\frac{I}{J}\right)$ by $\left(\frac{I}{J}\right) = \left(\frac{mm_E}{J}\right)$. If I is n^{th}-power-free, we denote by χ_I the character $\chi_I(J) = \left(\frac{I}{J}\right)$. This character depends on the choices above, but we suppress this from the notation.

Proposition 2.1 (Reciprocity)**.** [6] *Let I, $J \in \mathcal{J}(S)$ be coprime, and $\alpha(I,J) = \chi_I(J)\chi_J(I)^{-1}$. Then $\alpha(I,J)$ depends only on the images of I and J in R_C.*

Let $\mathcal{I}(S)$ denote the integral ideals prime to S_f. Let π be as above and let $L_S(s, \pi\otimes\chi_J)$ be the L-function for π twisted by the character χ_J, with the places in S removed. (Note that the Euler factor is also 1 at the places dividing J.) If ξ is any idèle class character then the twisted L-function $L(s, \pi\otimes\xi)$ satisfies a functional equation

$$L(s,\pi\otimes\xi) = \epsilon(s,\pi\otimes\xi)\, L(1-s,\tilde{\pi}\otimes\xi^{-1}), \tag{2.1}$$

where $\epsilon(s,\pi\otimes\xi)$ is the epsilon factor of $\pi\otimes\xi$.

Proposition 2.2. *Let $E, J \in \mathcal{O}(S)$ be n^{th}-power-free with associated characters χ_E, χ_J of conductors $\mathfrak{f}_E, \mathfrak{f}_J$ respectively. Suppose that $\chi_J = \chi_E\chi_I$ with $I \in K^\times$, $I \equiv 1 \bmod C$. Then*

$$\epsilon(s,\pi\otimes\chi_J) = \epsilon(1/2,\chi_I)^2\,\chi_\pi(\mathfrak{f}_J/\mathfrak{f}_E)\,(\mathbb{N}\mathfrak{f}_J/\mathbb{N}\mathfrak{f}_E)^{2(1/2-s)}\,\epsilon(s,\pi\otimes\chi_E). \tag{2.2}$$

Here $\epsilon(1/2, \chi_I)$ is given by a (normalized) n^{th} order Gauss sum, as in Tate's thesis. We henceforth assume that π has trivial central character (and is self-contragredient). Let

$$L_S(s,\pi) = \prod_{v \notin S} (1-\alpha_v q_v^{-s})^{-1}\,(1-\beta_v q_v^{-s})^{-1} = \sum_{I \in \mathcal{I}(S)} \frac{a(I)}{(\mathbb{N}I)^s},$$

where α_v and β_v are the Satake parameters associated to π at v. For J in $\mathcal{I}(S)$, write $J = J_0 J_n^n$, with J_0 the n^{th} power free part of J. For I in $\mathcal{I}(S)$, let $\widetilde{I}$ represent the part of I coprime to J_0.

For ideals I and J, define the function $G(I,J)$ by

$$G(I,J) = \prod_{\substack{v \\ \text{ord}_v(I)=\alpha \\ \text{ord}_v(J)=\beta}} G(P_v^\alpha, P_v^\beta),$$

where, for $\alpha, \beta \geq 0$,

$$G(P_v^\alpha, P_v^\beta) = \begin{cases} 1 & \text{if } \beta = 0 \\ q_v^{\beta/2-1}(q_v-1) & \text{if } \alpha \geq \beta, \quad \beta \equiv 0(n), \quad \beta > 0 \\ -q_v^{\beta/2-1} & \text{if } \alpha = \beta-1, \quad \beta \equiv 0(n), \quad \beta > 0 \\ q_v^{(\beta-1)/2} & \text{if } \alpha = \beta-1, \quad \beta \not\equiv 0(n), \quad \beta > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

To simplify notation, let ζ denote the Dedekind zeta function of K and ζ_S the zeta function with the places in S removed.

Define the following pair of multiple Dirichlet series:

$$Z_1(s,w;\pi,\psi_1,\psi_2) = \zeta_S(nw - n/2 + 1) \sum_{I,J \in \mathcal{I}(S)} \frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\overline{\chi}_{J_0}(\widetilde{I})\epsilon(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w}, \tag{2.4}$$

and

$$Z_2(s,w;\pi,\psi_1,\psi_2) = \zeta_S(nw - n/2 + 1) \sum_{I,J \in \mathcal{I}(S)} \frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\chi_{J_0}(\widetilde{I})\overline{\epsilon}(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w}, \tag{2.5}$$

where ψ_1, ψ_2 are two idèle class characters of R_C (hence of order dividing n and conductor dividing C). The notation $\epsilon(J_0)$ simply abbreviates $\epsilon(\frac{1}{2}, \chi_{J_0})$. (Note that Z_1 and Z_2 are essentially dual objects up to conjugation of χ_{J_0} and $\epsilon(J_0)$.)

By summing first over J, we have

$$Z_1(s,w;\pi,\psi_1,\psi_2) = \sum_{I \in \mathcal{I}(S)} \frac{a(I)\psi_1(I)D(w,I,\psi_2)}{(\mathbb{N}I)^s}, \tag{2.6}$$

where

$$D(w,I,\psi_2)=\zeta_S(nw-n/2+1)\sum_{J\in\mathcal{I}(S)}\frac{\psi_2(J)G(I,J)\overline{\chi}_{J_0}(\widetilde{I})\epsilon(J_0)}{(\mathbb{N}J)^w} \tag{2.7}$$

is a Dirichlet series obtained from the Fourier coefficient of an Eisenstein series defined on an appropriately restricted congruence subgroup Γ of the n-fold cover of GL(2). (This motivates the definition (2.3).) These metaplectic Eisenstein series were first formulated by Kubota (cf. [14]) and were studied in further detail by Kazhdan and Patterson in [11]. In particular, Kazhdan and Patterson exhibited a functional equation as $w\mapsto 1-w$ and determined the polar structure: $D(w,I,\psi_2)$ has possible simple poles at $w=\frac{1}{2}\pm\frac{1}{n}$ and is holomorphic elsewhere. (For a general introduction to the subject, we refer the reader to [10].)

Using the above theory together with one-variable convexity results, one sees that for every $\epsilon>0$

$$\left|(w-\tfrac{1}{2}+\tfrac{1}{n})(w-\tfrac{1}{2}-\tfrac{1}{n})D(w,I,\psi_2)\right|\ll_\epsilon \max\{1,(\mathbb{N}I)^{(1-\mathrm{Re}(w))/2+\epsilon},(\mathbb{N}I)^{1/2-\mathrm{Re}(w)+\epsilon}\}. \tag{2.8}$$

Hence, we can obtain a region of absolute convergence for our multiple Dirichlet series as a convolution of a GL(2) automorphic L-series and the above series $D(w,I,\psi_2)$. That is, for $i=1,2$ we define

$$\widetilde{Z}_i(s,w;\pi,\psi_1,\psi_2)=s(1-s)(w+\tfrac{1}{2}-\tfrac{1}{n})(w+\tfrac{1}{2}-\tfrac{1}{n})Z_i(s,w;\pi,\psi_1,\psi_2), \tag{2.9}$$

and it follows from (2.8) that $\widetilde{Z}_i(s,w;\pi,\psi_1,\psi_2)$ for $i=1,2$ converges absolutely and uniformly on compacta (and uniformly bounded away from the boundary) for the region

$$\mathcal{R}'=\{(s,w)|\,\mathrm{Re}(s)>\max\{\tfrac{10}{9},\tfrac{29}{18}-\tfrac{\mathrm{Re}(w)}{2},\tfrac{29}{18}-\mathrm{Re}(w)\}\ \}. \tag{2.10}$$

This is demonstrated carefully in Section 5.

Our multiple Dirichlet series have another fruitful interpretation upon interchanging the order of summation, so that the inner sum is over ideals $I\in\mathcal{I}(S)$. To present this form, first define the *correction polynomials* $Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1)$, for ideals $J=\prod_v P_v^{\mathrm{ord}_v(J)}$, by

$$Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1)=\prod_{\substack{v\\ \mathrm{ord}_v(J)=n\gamma}}\mathcal{Q}(s,P_v^{n\gamma},\overline{\chi}_{J_0}\psi_1;\pi)\prod_{k=1}^{n-1}\prod_{\substack{v\\ \mathrm{ord}_v(J)=n\gamma+k}}\left(\frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{k-1})}{q_v^{(n\gamma+k-1)s-\frac{n\gamma+k-1}{2}}}\right), \tag{2.11}$$

where

$$\begin{aligned}\mathcal{Q}(s,P_v^{n\gamma},\chi;\pi)=a(P_v^{n\gamma})q_v^{\frac{n\gamma}{2}-n\gamma s}-a(P_v^{n\gamma-1})\chi(P_v)q_v^{\frac{n\gamma}{2}-(n\gamma+1)s}\\ -a(P_v^{n\gamma-1})\overline{\chi}(P_v)q_v^{\frac{n\gamma}{2}-1-(n\gamma-1)s}+a(P_v^{n\gamma-2})q_v^{\frac{n\gamma}{2}-1-n\gamma s},\end{aligned} \tag{2.12}$$

and where we make the convention that $a(x) = 0$ for all non-integral x. Then we will show the following result in Section 3.

Proposition 2.3. *In the region of absolute convergence $\mathcal{R}'$ given in* (2.10)*,*

$$Z_1(s,w,\pi;\psi_1,\psi_2) = \zeta_S(nw-n/2+1)\sum_{J\in\mathcal{I}(S)} \frac{\epsilon(J_0)\psi_2(J)L_S(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J)^w}\, Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1), \tag{2.13}$$

and

$$Z_2(s,w,\pi;\psi_1,\psi_2) = \zeta_S(nw-n/2+1)\sum_{J\in\mathcal{I}(S)} \frac{\overline{\epsilon}(J_0)\psi_2(J)L_S(s,\pi\otimes\chi_{J_0}\psi_1)}{(\mathbb{N}J)^w}\, Q(s,J;\pi,\chi_{J_0}\psi_1). \tag{2.14}$$

Given the results of Proposition 2.3, we can use upper bounds on the Fourier coefficients of our L-series and the finite Dirichlet polynomials $\mathcal{Q}$, together with standard one-variable convexity arguments, to show that the functions $\widetilde{Z}_i(s,w,\pi;\psi_1,\psi_2)$ for $i=1,2$ converge absolutely and uniformly on compacta in the region

$$\mathcal{R}'' = \{(s,w)|\,\mathrm{Re}(w) > \max\{1, \tfrac{19}{9} - \mathrm{Re}(s), 2-2\,\mathrm{Re}(s)\}\ \}.$$

Since $\mathcal{R}'$ and $\mathcal{R}''$ have a non-empty intersection, we see that the functions $\widetilde{Z}_i(s,w,\pi;\psi_1,\psi_2)$ for $i=1,2$ converge absolutely and uniformly on compacta in their union, given by

$$\mathcal{R} = \{(s,w)|\,\mathrm{Re}(w) > \max\{2-2\,\mathrm{Re}(s), \tfrac{19}{9} - \mathrm{Re}(s), \tfrac{29}{9} - \tfrac{\mathrm{Re}(s)}{2}, \tfrac{29}{18} - \mathrm{Re}(s)\}\ \}. \tag{2.15}$$

We use the two interpretations of the multiple Dirichlet series to exhibit functional equations as $w\mapsto 1-w$ and $s\mapsto 1-s$. Translating the region $\mathcal{R}$ under these equations will lead to an analytic continuation. We begin with the form of the series as a sum of metaplectic Eisenstein series as written in (2.6). By adding in the contributions at the infinite places, we can state a precise formulation of the functional equation inherited by the multiple Dirichlet series from the Eisenstein series. Define

$$\Gamma_n(w) \stackrel{\text{def}}{=} \left[(2\pi)^{-1/2} n^{nw-\frac{n}{2}+1}\right]^{r_2} \prod_{i=1}^{n-1}\Gamma(w-\frac{1}{2}+\frac{i}{n})\,|D_K|^{nw-\frac{n}{2}+1}, \tag{2.16}$$

where D_K denotes the discriminant of the field K and r_2 is the number of pairs of complex embeddings. This set of gamma factors comes directly from the Fourier analysis and multiplication formula for the gamma function. Then $\Gamma_n(w)D(w,I,\psi_2)$ has a functional equation as $w\mapsto 1-w$, which we exploit to obtain the following proposition.

Proposition 2.4. *In the region of absolute convergence $\mathcal{R}$ given in* (2.15),

$$\Gamma_n(1-w)Z_1(s+w-1/2,1-w;\pi,\psi_1,\psi_2)\prod_{v\in S_f}(1-q_v^{n/2-1-nw})=\sum_{\xi}\Phi(w,\psi_2,\xi)\Gamma_n(w)Z_1(s,w;\pi,\psi_1\psi_2\xi,\overline{\psi}_2),$$

where each $\Phi(w,\psi_2,\xi)$ is a function of w which is bounded in vertical strips of bounded width. The sum is taken over all characters ξ with conductor dividing C and order dividing n.

Proof: This follows as an immediate corollary of the functional equation for $\Gamma_n(w)D(w,I,\psi_2)$ given as Corollary II.2.4 of [11]. Note the necessity of twisting by characters ψ_1 and ψ_2 so that as the functional equation takes Eisenstein series to a linear combination of Eisenstein series at each cusp, the form of our basic Dirichlet series remains the same. Our $\Phi(w,\xi)$ is then essentially a scattering matrix for this functional equation. □

Now equipped with one functional equation, we go in search of a second. By interchanging the order of summation, decomposing the sums in (2.4) and (2.5) according to primes dividing J, we will view our Dirichlet series as weighted sums of L-series in s associated to π. Thus our series inherit functional equations as $s\mapsto 1-s$. To make this precise, we must first include the appropriate Gamma factors which complete the L-series. Define $\Gamma_K(s)$ by

$$\Gamma_K(s)=\left(\frac{|D_K|}{(2\pi)^{r_2}}\right)^s\Gamma(s+i\nu)^{r_2}\Gamma(s-i\nu)^{r_2}, \tag{2.17}$$

where again D_K is the discriminant of K, r_2 denotes the number of pairs of complex embeddings in our totally complex field K, and $\frac{1}{4}+\nu^2$ is the eigenvalue corresponding to the automorphic representation π.

Proposition 2.5. *In the region of absolute convergence $\mathcal{R}$,*

$$\begin{aligned}\prod_{v\in S_f}\left(1-\frac{\alpha_v\overline{\psi}_1(P_v^n)}{q_v^{n-ns}}\right)\left(1-\frac{\beta_v\overline{\psi}_1(P_v^n)}{q_v^{n-ns}}\right)\Gamma_K(s)\zeta_S(nw+2ns-\tfrac{3n}{2}+1)Z_1(s,w,\pi,\psi_1,\psi_2)\\ =\sum_{\xi\in\hat{R}_C}B(s;\psi_1,\xi)\Gamma_K(1-s)\zeta_S(nw-\tfrac{n}{2}+1)Z_2(1-s,w+2s-1;\pi,\psi_1,\psi_2\psi_1^2\xi)\end{aligned} \tag{2.18}$$

where the product of functions $\Gamma_K(1-s)B(s;\psi_1,\xi)$ is bounded in vertical strips of bounded width.

The proof is completed in Section 4. Because we will apply both functional equations to our multiple Dirichlet series in order to obtain an analytic continuation, we would like to define Γ-factors at the infinite places which are invariant under both functional equations as $s \mapsto 1-s$ and $w \mapsto 1-w$. Thus, according to the previous propositions, we define

$$\Gamma(s,w) = \Gamma_K(s)\Gamma_K(s+w-1/2)\Gamma_n(w)\Gamma_n(w+2s-1), \tag{2.19}$$

and define

$$Z_i^*(s,w;\pi,\psi_1,\psi_2) = \Gamma(s,w)\zeta_S(nw+2ns-\tfrac{3n}{2}+1)Z_i(s,w;\pi,\psi_1,\psi_2) \quad \text{for } i=1,2. \tag{2.20}$$

The pair of functional equations from Propositions 2.4 and 2.5, repeatedly applied to a region of absolute convergence, provide an analytic continuation to all of $\mathbb{C}^2$. This is demonstrated in Section 5. We will show in Section 6 that the resulting expression is uniformly convergent in some right half-plane. The functional equation together with this convergence will allow us to show in Section 7 that if there is a single nonvanishing twist at $s=\frac{1}{2}$, there must in fact be infinitely many nonvanishing twists at $s=\frac{1}{2}$.

3. Interchanging the Order of Summation – A Proof of Proposition 2.3

Proof of Proposition 2.3: We give the proof for (2.13), noting that the proof of (2.14) follows identically. Fix an ideal J in $\mathcal{I}(S)$ and decompose it according to $J = J_0 J_n^n$ where $J_0 = J_1 J_2^2 \dots J_{n-1}^{n-1}$ again denotes the n^{th}-power free part of J. Let $v \notin S$ be a place such that $\text{ord}_v(J) = n\gamma$, so that we may write $J = P_v^{n\gamma} J'$ with $(J', P_v) = 1$. We must analyze the resulting object $G(I,J)$.

Writing $I = P_v^{\lambda} I'$ with $(I', P_v) = 1$, gives

$$\sum_{I \in \mathcal{I}(S)} \frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\overline{\chi}_{J_0}(\widetilde{I})\epsilon(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w} = \sum_{\substack{I' \\ \text{ord}_v(I')=0}} \frac{a(I')\psi_1(I')\psi_2(J)G(I',J')\overline{\chi}_{J_0}(\widetilde{I'})\epsilon(J_0)}{(\mathbb{N}I')^s(\mathbb{N}J')^w} \times$$
$$\sum_{\lambda \geq 0} \frac{a(P_v^{\lambda})\psi_1(P_v^{\lambda})G(P_v^{\lambda}, P_v^{n\gamma})\overline{\chi}_{J_0}(P_v^{\lambda})}{(\mathbb{N}P_v)^{\lambda s + n\gamma w}}.$$

We first evaluate the sum over λ. If $\gamma = 0$, the sum becomes

$$\sum_{\lambda \geq 0} \frac{a(P_v^{\lambda})\psi_1(P_v^{\lambda})\overline{\chi}_{J_0}(P_v^{\lambda})}{(\mathbb{N}P_v)^{\lambda s}} = L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0}\psi_1),$$

where $L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0}\psi_1)$ denotes the Euler factor associated to the place v in the L-series.

If $\gamma \geq 1$, then using (2.3) we obtain

$$\sum_{\lambda \geq 0} \frac{a(P_v^\lambda)\psi_1(P_v^\lambda)G(P_v^\lambda, P_v^{n\gamma})\overline{\chi}_{J_0}(P_v^\lambda)}{q_v^{\lambda s + n\gamma w}} = -\frac{a(P_v^{n\gamma-1})\psi_1(P_v^{n\gamma-1})\overline{\chi}_{J_0}(P_v^{n\gamma-1})}{q_v^{n\gamma w + (n\gamma-1)s - \frac{n\gamma}{2}+1}} + \frac{1}{q_v^{n\gamma w - \frac{n\gamma}{2}+1}} \sum_{\lambda \geq n\gamma} \frac{(q_v-1)a(P_v^\lambda)\psi_1(P_v^\lambda)\overline{\chi}_{J_0}(P_v^\lambda)}{q_v^{\lambda s}}. \tag{3.1}$$

We wish to sum this geometric series, pulling out a factor of $L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0}\psi_1)$. We must therefore write the Fourier coefficients $a(P_v^\lambda)$ in terms of the v^{th} Satake parameters. (Recall that for $v \notin S$, $\alpha_v \beta_v = 1$ and $\alpha_v + \beta_v = a(P_v)$.) We have

$$a(P_v^\lambda) = \frac{\alpha_v^{\lambda+1} - \beta_v^{\lambda+1}}{\alpha_v - \beta_v},$$

and

$$\begin{aligned} L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0}\psi_1) &= \left(1 - \frac{\alpha_v \overline{\chi}_{J_0}(P_v)\psi_1(P_v)}{q_v^s}\right)^{-1} \left(1 - \frac{\beta_v \overline{\chi}_{J_0}(P_v)\psi_1(P_v)}{q_v^s}\right)^{-1} \\ &= (1 - a(P_v)\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{-s} + \chi_{J_0}(P_v)\overline{\psi}_1(P_v)q_v^{-2s})^{-1}. \end{aligned}$$

Substituting these definitions into the latter sum of (3.1) and evaluating the geometric sums, we have

$$\begin{aligned} \sum_{\lambda \geq n\gamma} \frac{a(P_v^\lambda)\psi_1(P_v^\lambda)\overline{\chi}_{J_0}(P_v^\lambda)}{q_v^{\lambda s}} &= \frac{1}{\alpha_v - \beta_v}\left[\sum_{\lambda \geq n\gamma} \frac{\alpha_v^{\lambda+1}\overline{\chi}_{J_0}(P_v^\lambda)\psi_1(P_v^\lambda)}{q_v^{\lambda s}} - \sum_{\lambda \geq n\gamma} \beta_v^{\lambda+1}\frac{\overline{\chi}_{J_0}(P_v^\lambda)\psi_1(P_v^\lambda)}{q_v^{\lambda s}}\right] \\ &= \frac{1}{\alpha_v - \beta_v}\left[\frac{\alpha_v^{n\gamma+1}}{q_v^{n\gamma s}}(1 - \alpha_v\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{-s})^{-1} - \frac{\beta_v^{n\gamma+1}}{q_v^{n\gamma s}}(1 - \beta_v\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{-s})^{-1}\right] \\ &= \frac{L^{(v)}(s, \pi, \overline{\chi}_{J_0}\psi_1)}{q_v^{n\gamma s}}\big(a(P_v^{n\gamma}) - a(P_v^{n\gamma-1})\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{-s}\big). \end{aligned}$$

Therefore, we may rewrite the entire equation (3.1) as

$$\begin{aligned} &\sum_{\lambda \geq 0} \frac{a(P_v^\lambda)\psi_1(P_v^\lambda)G(P_v^\lambda, P_v^{n\gamma})\overline{\chi}_{J_0}(P_v^\lambda)}{q_v^{\lambda s + n\gamma w}} \\ &\quad = \frac{1}{q_v^{n\gamma w + n\gamma s - \frac{n\gamma}{2}+1}}\big[-a(P_v^{n\gamma-1})\overline{\psi}_1(P_v)\chi_{J_0}(P_v)\, q_v^s \\ &\qquad + (q_v - 1)\left[a(P_v^{n\gamma}) - a(P_v^{n\gamma-1})\psi_1(P_v)\overline{\chi}_{J_0}(P_v)q_v^{-s}\right] L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0}\psi_1)\big] \\ &= \frac{L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0}\psi_1)}{q_v^{n\gamma w + n\gamma s - \frac{n\gamma}{2}+1}} \cdot \big(-a(P_v^{n\gamma-1})\overline{\psi}_1(P_v)\chi_{J_0}(P_v)\, q_v^s\left[1 - a(P_v)\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{-s}\right. \\ &\qquad \left. + \chi_{J_0}(P_v)\overline{\psi}_1(P_v)q_v^{-2s}\right] + (q_v - 1)\left[a(P_v^{n\gamma}) - a(P_v^{n\gamma-1})\psi_1(P_v)\overline{\chi}_{J_0}(P_v)q_v^{-s}\right]\big). \end{aligned}$$

Expanding in the second bracket, and using the relation

$$a(P_v^{n\gamma-1})a(P_v) = (\alpha_v - \beta_v)^{-1}(\alpha_v^{n\gamma} - \beta_v^{n\gamma})(\alpha_v + \beta_v) = a(P_v^{n\gamma}) + a(P_v^{n\gamma-2}),$$

we find that the above expression equals

$$\frac{L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)}{q_v^{n\gamma w+n\gamma s-\frac{n\gamma}{2}+1}}\left[q_v a(P_v^{n\gamma})-a(P_v^{n\gamma-1})\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{1-s}\right.$$
$$\left.-a(P_v^{n\gamma-1})\chi_{J_0}(P_v)\overline{\psi}_1(P_v)q_v^{s}+a(P_v^{n\gamma-1})a(P_v)\right].$$

Putting it all together, we have

$$\sum_{\lambda\geq 0}\frac{a(P_v^{\lambda})\psi_1(P_v^{\lambda})G(P_v^{\lambda},P_v^{n\gamma})\overline{\chi}_{J_0}(P_v^{\lambda})}{q_v^{\lambda s+n\gamma w}}=$$
$$=\frac{L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)}{q_v^{n\gamma w}}\Big[a(P_v^{n\gamma})q_v^{\frac{n\gamma}{2}-n\gamma s}-a(P_v^{n\gamma-1})\overline{\chi}_{J_0}(P_v)\psi_1(P_v)q_v^{\frac{n\gamma}{2}-(n\gamma+1)s}-$$
$$a(P_v^{n\gamma-1})\chi_{J_0}(P_v)\overline{\psi}_1(P_v)q_v^{\frac{n\gamma}{2}-1-(n\gamma-1)s}+a(P_v^{n\gamma-2})q_v^{\frac{n\gamma}{2}-1-n\gamma s}\Big].$$

Recall the convention that $a(x)=0$ for all non-integral x. Then repeating the above process for all such places v with P_v not dividing J_0, we have for any fixed ideal J

$$\sum_{I}\frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\overline{\chi}_{J_0}(\widetilde{I})\epsilon(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w}\tag{3.2}$$
$$=\prod_{\substack{v\\ \mathrm{ord}_v(J)=n\gamma}}\frac{L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)}{q_v^{n\gamma w}}\mathcal{Q}(s,P_v^{n\gamma},\overline{\chi}_{J_0}\psi_1;\pi)\sum_{\substack{I\\ I|J_0^\infty}}\frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\overline{\chi}_{J_0}(\widetilde{I})}{(\mathbb{N}I)^s(\mathbb{N}J_0)^w},$$

where $\mathcal{Q}(s,P_v^{n\gamma},\chi_{J_0}\psi_1;\pi)$ is as defined in (2.12).

We must now repeat this analysis for the remaining sum over I such that $I|J_0^\infty$. Let v be a place such that $P_v|J_0$. That is, $\mathrm{ord}_v(J)=n\gamma+k$, for some $k\in\{1,2,\dots n-1\}$ and denote $\mathrm{ord}_v(I)=\lambda$. Then, writing $I=P_v^{\lambda}I'$ and $J=P_v^{n\lambda+k}J'$, we have

$$G(I,J)=G(P_v^{\lambda}I',P_v^{n\gamma+k}J')=\begin{cases}q_v^{\frac{n\gamma+k-1}{2}}G(I',J'), & \text{if } \lambda=n\gamma+k-1\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, in this case $\widetilde{P_v}=(1)$ since $P_v|J_0$, so $\chi_{J_0}(\widetilde{P_v})=1$ Thus we may write

$$\sum_{\substack{I\\ I|J_0^\infty}}\frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\epsilon(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w}=\tag{3.3}$$
$$\left(\frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{n\gamma+k-1})}{q_v^{(n\gamma+k-1)s+(n\gamma+k)w-\frac{n\gamma+k-1}{2}}}\right)\sum_{\substack{I'\\ I'|J_0^\infty}}\frac{a(I')\psi_1(I')\psi_2(J)G(I',J')\overline{\chi}_{J_0}(\widetilde{I'})\epsilon(J_0)}{(\mathbb{N}I')^s(\mathbb{N}J')^w}.$$

Repeating this for the remaining finite list of places v such that $P_v|J_0$, we have

$$\sum_{\substack{I\\ I|J_0^\infty}}\frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\epsilon(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w}=\epsilon(J_0)\psi_2(J)\prod_{k=1}^{n-1}\prod_{\substack{v\\ \mathrm{ord}_v(J)=n\gamma+k}}\left(\frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{n\gamma+k-1})}{q_v^{(n\gamma+k-1)s+(n\gamma+k)w-\frac{n\gamma+k-1}{2}}}\right).\tag{3.4}$$

Combining this result with the information from (3.2) and (2.12) and noting that our characters ψ_i have order n, the original series for fixed J takes form

$$\sum_{I \in \mathcal{I}(S)} \frac{a(I)\psi_1(I)\psi_2(J)G(I,J)\overline{\chi}_{J_0}(\widetilde{I})\epsilon(J_0)}{(\mathbb{N}I)^s(\mathbb{N}J)^w} = \epsilon(J_0)\psi_2(J)L_S(s,\pi\otimes\overline{\chi}_{J_0}\psi_1) \times \tag{3.5}$$

$$\prod_{\substack{v \\ \mathrm{ord}_v(J)=n\gamma}} \frac{\mathcal{Q}(s,P_v^{n\gamma},\overline{\chi}_{J_0}\psi_1;\pi)}{q_v^{n\gamma w}} \prod_{k=1}^{n-1} \prod_{\substack{v \\ \mathrm{ord}_v(J)=n\gamma+k}} \left(\frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{k-1})}{q_v^{(n\gamma+k-1)s+(n\gamma+k)w-\frac{n\gamma+k-1}{2}}} \right).$$

Summing over each ideal $J \in \mathcal{I}(S)$, the result follows. □

We will also have need of the following lemma.

Lemma 3.1. *Let the notation be as above. The correction factor $Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1)$ satisfies the following functional equation in s:*

$$Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1) = (\mathbb{N}J_2J_3^2\cdots J_{n-1}^{n-2}J_n^n)^{1-2s}\psi^2(J_2J_3^2\cdots J_{n-1}^{n-2}J_n^n)Q(1-s,J;\pi,\chi_{J_0}\overline{\psi}_1). \tag{3.6}$$

Proof: From the definition made in (2.12), one readily sees that at each prime ideal P_v with $(P_v,J_0)=1$,

$$\mathcal{Q}(s,P_v^{n\gamma},\overline{\chi}_{J_0}\psi_1;\pi) = (q_v^{n\gamma})^{1-2s}\mathcal{Q}(1-s,P_v^{n\gamma},\chi_{J_0}\overline{\psi}_1;\pi).$$

Moreover, for each of the prime ideals P_v with $P_v|J_0$ for any choice of k, we have the identity

$$\left(\frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{k-1})}{q_v^{(n\gamma+k-1)s-\frac{n\gamma+k-1}{2}}} \right) = (q_v^{n\gamma+k-1})^{1-2s}\psi_1^2(P_v^{k-1}) \left(\frac{a(P_v^{n\gamma+k-1})\overline{\psi}_1(P_v^{k-1})}{q_v^{(n\gamma+k-1)(1-s)-\frac{n\gamma+k-1}{2}}} \right).$$

The lemma therefore follows by combining the above identities. □

4. A Functional Equation as $s \mapsto 1-s$: A Proof of Proposition 2.5

Recall from Proposition 2.3 that we have

$$Z_1(s,w,\pi;\psi_1,\psi_2) = \zeta_S(nw-n/2+1) \sum_{J\in\mathcal{I}(S)} \frac{\epsilon(J_0)\psi_2(J)L_S(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J)^w} Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1),$$

and

$$Z_2(s,w,\pi;\psi_1,\psi_2) = \zeta_S(nw-n/2+1) \sum_{J\in\mathcal{I}(S)} \frac{\overline{\epsilon}(J_0)\psi_2(J)L_S(s,\pi\otimes\chi_{J_0}\psi_1)}{(\mathbb{N}J)^w} Q(s,J;\pi,\chi_{J_0}\psi_1),$$

where $Q(s,J;\pi,\overline{\chi}_{J_0}\psi_1)$ is the correction polynomial defined in (2.11). To facilitate the statement of the results of this section, we extend the definitions of Z_1 and Z_2 to include arbitrary linear combinations of characters in place of ψ_1 and ψ_2. In particular for $E \in \mathcal{E}$, let δ_E be the characteristic function of the class E, and consider

$$Z_1(s, w; \pi, \psi_1, \delta_E \psi_2) = \zeta_S(nw - n/2 + 1) \sum_{\substack{J \in \mathcal{I}(S) \\ J \sim E}} \frac{\epsilon(1/2, J_0)\psi_2(J) L_S(s, \pi \otimes \overline{\chi}_J \psi_1)}{(\mathbb{N}J)^w}, \tag{4.1}$$

where for notational convenience, we put

$$L(s, \pi \otimes \overline{\chi}_J \psi_1) = L(s, \pi \otimes \overline{\chi}_{J_0} \psi_1)\, Q(s, J; \pi, \overline{\chi}_{J_0} \psi_1). \tag{4.2}$$

We determine the functional equation for this completed L-function in the following lemma.

Lemma 4.1. *With the notation as above,*

$$L(s, \pi \otimes \overline{\chi}_J \psi_1) = \varepsilon(\tfrac{1}{2}, \overline{\chi}_{I_0})^2\, \epsilon(s, \pi \otimes \overline{\chi}_E \psi_1)\, \psi_1^2 \left(\frac{JC_E}{\mathfrak{f}_E}\right) \left(\frac{\mathbb{N}JC_E}{\mathbb{N}\mathfrak{f}_E}\right)^{1-2s} L(1 - s, \pi \otimes \chi_J \overline{\psi}_1). \tag{4.3}$$

Proof: From (2.1), we have $L(s, \pi \otimes \overline{\chi}_{J_0} \psi_1) = \varepsilon(s, \pi \otimes \overline{\chi}_{J_0} \psi_1)\, L(1 - s, \pi \otimes \chi_{J_0} \overline{\psi}_1)$. We will evaluate the factor $\varepsilon(s, \pi \otimes \overline{\chi}_{J_0} \psi_1) = \varepsilon(s, (\pi \otimes \psi_1) \otimes \overline{\chi}_{J_0})$ using Proposition 2.2. Write $J_0 = I_0 E$, where E represents the class of J_0 in R_C, and where $I_0 \equiv 1 \mod C$. The conductor of χ_{J_0} is given by $\mathfrak{f}_{J_0} = J_1 J_2 \cdots J_{n-1} C_E$, where C_E is a constant depending only on the class E. The central character of π is trivial, therefore the central character of $\pi \otimes \psi_1$ is ψ_1^2. Thus we have

$$\varepsilon(s, \pi \otimes \overline{\chi}_{J_0} \psi_1) = \varepsilon(\tfrac{1}{2}, \overline{\chi}_{I_0})^2\, \psi_1^2 \left(\frac{J_1 \cdots J_{n-1} C_E}{\mathfrak{f}_E}\right) \left(\frac{\mathbb{N}J_1 \cdots J_{n-1} C_E}{\mathbb{N}\mathfrak{f}_E}\right)^{1-2s} \varepsilon(s, \pi \otimes \psi_1 \overline{\chi}_E) \tag{4.4}$$

The lemma then follows by combining the above with the functional equation for $Q(s, J; \pi, \overline{\chi}_{J_0} \psi_1)$ given in Lemma 3.1. □

We are now ready to demonstrate the functional equation for Z_1 as $s \mapsto 1 - s$. The functional equation in Z_2 can be shown completely analogously.

Proof of Proposition 2.5: Using Lemma 4.1, together with the fact that

$$\varepsilon(\frac{1}{2}, \overline{\chi}_{I_0})^2\, \varepsilon(\frac{1}{2}, \chi_{J_0}) = \varepsilon(\frac{1}{2}, \overline{\chi}_{J_0})\, \varepsilon(\frac{1}{2}, \chi_E)^2,$$

we see that

$$\begin{aligned} &Z_1(s, w; \pi, \psi_1, \delta_E \psi_2) \\ &\quad = \mathcal{A}(E, s) \sum_{\substack{J \in \mathcal{I}(S) \\ J \sim E}} \frac{\overline{\epsilon}(1/2, J_0)\psi_2 \psi_1^2(J) L_S(1 - s, \pi \otimes \chi_{J_0} \overline{\psi}_1)}{(\mathbb{N}J)^{w+2s-1}} \prod_{v \in S} \frac{L^{(v)}(1 - s, \pi \otimes \chi_{J_0} \overline{\psi}_1)}{L^{(v)}(s, \pi \otimes \overline{\chi}_{J_0} \psi_1)}, \end{aligned} \tag{4.5}$$

where

$$\mathcal{A}(E,s) = \left(\frac{\mathbb{N}C_E}{\mathbb{N}\mathfrak{f}_E}\right)^{1-2s} \psi_1^2\left(\frac{C_E}{\mathfrak{f}_E}\right)\epsilon(1/2,\chi_E)^2\epsilon(s,\pi\otimes\psi_1\overline{\chi}_E).$$

To proceed further, multiply both sides of (4.5) by $\prod_{v\in S_f}\frac{1}{L^{(v)}(n-ns,\pi\otimes\overline{\psi_1^n})}$. Then

$$\frac{L^{(v)}(1-s,\pi\otimes\chi_{J_0}\overline{\psi}_1)}{L^{(v)}(n-ns,\pi\otimes\overline{\psi_1^n})} = \left[1+\frac{\alpha_v\chi_{J_0}\overline{\psi}_1(P_v)}{q_v^{1-s}}+\cdots+\frac{\alpha_v^{n-1}\chi_{J_0}\overline{\psi}_1(P_v^{n-1})}{q_v^{(n-1)(1-s)}}\right] \times\left[1+\frac{\beta_v\chi_{J_0}\overline{\psi}_1(P_v)}{q_v^{1-s}}+\cdots+\frac{\beta_v^{n-1}\chi_{J_0}\overline{\psi}_1(P_v^{n-1})}{q_v^{(n-1)(1-s)}}\right].$$

Hence for each J and each $v\in S_f$ the term

$$\frac{L^{(v)}(1-s,\pi\otimes\chi_{J_0}\overline{\psi}_1)}{L^{(v)}(n-ns,\pi\otimes\overline{\psi_1^n})}\frac{1}{L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)}$$

becomes a finite Laurent polynomial in q_v^s whose dependence on J is through terms of the form $\chi_{J_0}(P_v)$. Since $v\in S_f$ and J is in a fixed class E of R_C, we have $\chi_{J_0}(P_v)=\xi_v(E)$ for some character ξ_v of R_C. Similarly, the quotient

$$\prod_{v\in S_\infty}\frac{L^{(v)}(1-s,\pi\otimes\chi_{J_0}\overline{\psi}_1)}{L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)} = \frac{\Gamma_K(1-s)}{\Gamma_K(s)}, \tag{4.6}$$

where $\Gamma_K(s)$ is defined as in (2.17), is independent of J.

Since

$$Z_1^*(s,w;\pi,\psi_1,\psi_2) = \Gamma(s,w)\zeta_S(nw+2ns-\tfrac{3n}{2}+1)Z_1(s,w;\pi,\psi_1,\psi_2),$$

where we recall that $\Gamma(s,w)$ is the complete set of Gamma factors defined in (2.19), we conclude that

$$\prod_{v\in S_f}\left(1-\frac{\alpha_v\overline{\psi}_1(P_v^n)}{q_v^{n-ns}}\right)\left(1-\frac{\beta_v\overline{\psi}_1(P_v^n)}{q_v^{n-ns}}\right)Z_1^*(s,w,\pi,\psi_1,\psi_2\delta_E) = A(s;\psi_1,E)Z_2^*(1-s,w+2s-1;\pi,\psi_1,\psi_2\psi_1^2\delta_E).$$

Moreover, the functions $A(s;\psi_1,E)$ are finite Laurent polynomials in $\mathbb{N}J^s$. Summing over $\mathcal{E}$, we find that

$$\prod_{v\in S_f}\left(1-\frac{\alpha_v\overline{\psi}_1(P_v^n)}{q_v^{n-ns}}\right)\left(1-\frac{\beta_v\overline{\psi}_1(P_v^n)}{q_v^{n-ns}}\right)Z_1^*(s,w,\pi,\psi_1,\psi_2) = \sum_{E\in\mathcal{E}}Z_1^*(s,w,\pi,\psi_1,\psi_2\delta_E)$$

is equal to

$$\sum_{\xi}B(s;\psi_1,\xi)Z_2^*(1-s,w+2s-1;\pi,\psi_1,\psi_2\psi_1^2\xi)$$

where $B(s;\psi_1,\xi)$ is a linear combination of the $A(s;\psi_1,E)$. $\square$

5. Analytic Continuation

We wish to analytically continue the functions $\widetilde{Z}_i(s,w;\pi,\psi_1,\psi_2)$ for $i=1,2$ to $\mathbb{C}^2$, for each choice of ψ_1 and ψ_2. This will be achieved using the functional equations for the Z_i^*, along with properties of the series $D(w,I,\psi_2)$ and the Dirichlet series $L(s,\pi\otimes\overline{\chi}_d\psi_1)$. As above, we will restrict our attention to $\widetilde{Z}_1$, as the arguments for $\widetilde{Z}_2$ will be almost identical.

First, we consider the expression for $Z_1(s,w;\pi,\psi_1,\psi_2)$ given in (2.6). Using the bound $|a(I)| \ll_\epsilon (\mathbb{N}I)^{1/9+\epsilon}$ for the Fourier coefficients, as well as the bound given in (2.8), we see that $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$ converges absolutely and uniformly in the region of $\mathbb{C}^2$ satisfying $\mathrm{Re}(w)>1$ and $\mathrm{Re}(s)>\frac{10}{9}$.

Next, we examine the behavior of $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$ when $\mathrm{Re}(s)\le -1/9$, utilizing the expression for $Z_1(s,w;\pi,\psi_1,\psi_2)$ given in (2.13). It will be convenient to work with this series as

$$Z_1(s,w;\pi,\psi_1,\psi_2)=\sum_{E\in R_C} Z_1(s,w;\pi,\psi_1,\delta_E\psi_2),$$

with $Z_1(s,w;\pi,\psi_1,\delta_E\psi_2)$ as given in (4.1). The full Dirichlet series $L(s,\pi\otimes\overline{\chi}_J\psi_1)$ satisfies the functional equation given in (4.3). This functional equation involves gamma factors, as we have

$$L(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)=\prod_v L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)=\Gamma_K(s)A(s)L_S(s,\pi\otimes\overline{\chi}_{J_0}\psi_1), \tag{5.1}$$

where $A(s)=\prod_{v\in S_f} L^{(v)}(s,\pi\otimes\overline{\chi}_{J_0}\psi_1)$. Combining (5.1) with (4.3), we obtain

$$L_S(s,\pi\otimes\overline{\chi}_J\psi_1)=\frac{\Gamma_K(1-s)}{\Gamma_K(s)}\frac{A(1-s)}{A(s)}B(s,E)(\mathbb{N}J)^{1-2s}L_S(1-s,\pi\otimes\overline{\chi}_J\psi_1), \tag{5.2}$$

where we put

$$B(s,E)=\varepsilon(\tfrac{1}{2},\overline{\chi}_{I_0})^2\varepsilon(s,\pi\otimes\overline{\chi}_E\psi_1)\,\psi_1^2\left(\frac{JC_E}{\mathfrak{f}_E}\right)\left(\frac{\mathbb{N}C_E}{\mathbb{N}\mathfrak{f}_E}\right)^{1-2s}.$$

We set $s=-1/9+\sigma+it$ in (5.2) and examine the factors on the right, as $|t|\to\infty$. For the Gamma factors, using a simplified version of Stirling's formula given by

$$|\Gamma(\sigma+it)|\sim |t|^{\sigma-1/2}exp(-\pi|t|) \text{ as } |t|\to\infty,$$

we see that

$$\left|\frac{\Gamma_K(10/9-\sigma-it)}{\Gamma_K(-1/9+\sigma+it)}\right|\sim\left(\frac{|D_K|}{(2\pi)^{r_2}}|t^2-\nu^2|^{r_2}\right)^{11/9-2\sigma}$$

where ν is the constant given in (2.17). As $|t|\to\infty$, we therefore have

$$\left|\frac{\Gamma_K(10/9-\sigma-it)}{\Gamma_K(-1/9+\sigma+it)}\right|\sim M|t|^{(22/9-4\sigma)r_2}$$

for some positive constant M. The factor $\left|\frac{A(\frac{10}{9}-\sigma-it)}{A(\frac{-1}{9}+\sigma+it)}\right|$ is a finite polynomial in powers of q_v^σ, hence it is independent of t. Finally, the factor $L_S(\frac{10}{9}-\sigma-it,\pi\otimes\overline{\chi}_J\psi_1)$ is bounded as a function of t, since the full L-series is absolutely convergent when $s=\frac{10}{9}$. Thus we see that $L_S(-1/9+\sigma+it,\pi\otimes\overline{\chi}_J\psi_1)$ has polynomial growth as a function of t, for fixed $\sigma<0$.

For $\mathrm{Re}(s)\le -1/9$, we therefore have

$$|Z_1(s,w;\pi,\psi_1,\psi_2)| \ll_\epsilon c(\sigma)|t|^{(22/9-4\sigma)r_2} \sum_{E\in R_C}\sum_{\substack{J\in\mathcal{I}(S)\\ J\sim E}} \frac{1}{(\mathbb{N}J)^{\mathrm{Re}(w)+2\sigma-11/9+\epsilon}},$$

where $c(\sigma)$ is a constant independent of t, given by $c(\sigma)=\max_{E\in R_C}\{c(\sigma,E)\}$, with

$$c(\sigma,E)=\left|\frac{A(\frac{10}{9}-\sigma-it)}{A(\frac{-1}{9}+\sigma+it)}\right|\cdot\left|B(\tfrac{-1}{9}+\sigma+it)\right|\cdot M.$$

From this we see that $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$ converges absolutely and uniformly on compacta in the region satisfying $\mathrm{Re}(s)<-1/9$ and $\mathrm{Re}(w)>2-2\,\mathrm{Re}(s)$. Now by applying a generalized version of the Phragmén-Lindelöf Theorem, we see that for s with real part between $-1/9$ and $10/9$, $\widetilde{Z}_1$ is absolutely and uniformly convergent, provided $\mathrm{Re}(w)>19/9-\mathrm{Re}(s)$. Therefore, our region of convergence for $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$ is the region $\mathcal{R}''=\{(s,w)|\,\mathrm{Re}(w)>\max\{1,\frac{19}{9}-\mathrm{Re}(s),2-2\,\mathrm{Re}(s)\}\ \}$.

To obtain a second region of convergence for $\widetilde{Z}_1$, we again consider the expression for Z_1 in terms of $D(w,I,\psi_2)$, as given in (2.6). If $\mathrm{Re}(w)<0$, then (2.8) gives

$$|(w-\tfrac{1}{2}+\tfrac{1}{n})(w-\tfrac{1}{2}-\tfrac{1}{n})D(w,I,\psi_2)| \ll_\epsilon (\mathbb{N}I)^{1/2-\mathrm{Re}(w)+\epsilon},$$

hence we see that

$$|\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)| \ll_\epsilon \sum_{I\in\mathcal{I}(S)} \frac{(\mathbb{N}I)^{1/9+\epsilon}(\mathbb{N}I)^{1/2-\mathrm{Re}(w)}}{(\mathbb{N}I)^s} = \sum_{I\in\mathcal{I}(S)} \frac{1}{(\mathbb{N}I)^{s+\mathrm{Re}(w)-11/18+\epsilon}}.$$

Consequently, the initial region of convergence of $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$, i.e. $\mathrm{Re}(w)>1$ and $\mathrm{Re}(s)>\frac{10}{9}$, is extended to include the region of $\mathbb{C}^2$ satisfying $\mathrm{Re}(w)<0$ and $\mathrm{Re}(s)>29/18-\mathrm{Re}(w)$. Then by a second application of the Phragmén-Lindelöf theorem, we see that $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$ converges absolutely and uniformly on compact in the region $\mathcal{R}'=\{(s,w)|\,\mathrm{Re}(s)>\max\{\frac{10}{9},\frac{29}{18}-\frac{\mathrm{Re}(w)}{2},\frac{29}{18}-\mathrm{Re}(w)\}\ \}$.

These regions overlap, which means that $\widetilde{Z}_1(s,w;\pi,\psi_1,\psi_2)$ converges on their union, $\mathcal{R}=\mathcal{R}'\cup\mathcal{R}''$. By an almost identical argument, $\widetilde{Z}_2(s,w;\pi,\psi_1,\psi_2)$ also converges on the region $\mathcal{R}$. Now we may apply the functional equations for the Z_i, represented for convenience as $\alpha:(s,w)\to(1-s,w+2s-1)$ and

$\beta : (s,w) \to (s+w-1/2, 1-s)$, to extend this region of convergence. Applying the transformation α to the region of convergence $\mathcal{R}$, we obtain a region which overlaps $\mathcal{R}$, and when we take the convex hull of their union, we obtain the half-plane $\{(s,w) |\, \mathrm{Re}(s) > \frac{29}{18} - \mathrm{Re}(w)\}$.

Finally, applying the transformation β to this half-plane, we obtain another half-plane which overlaps it. Therefore when we take the convex hull of their union, we obtain all of $\mathbb{C}^2$, as desired.

6. Absolute Convergence of Sums in a Right Half-Plane

We first analyze the individual expressions $Z_i(\frac{1}{2}, w; \pi, \psi_1, \psi_2)$ for $i = 1, 2$. Again, we will restrict our attention to Z_1 in what follows, as the convergence of Z_2 will evidently follow in the same fashion. To begin, we separate the sum over J in terms of two pieces – the first corresponding to n^{th} power free J_0, and the second corresponding to J_n. From (2.13), we have

$$Z_1\left(\frac{1}{2}, w; \pi, \psi_1, \psi_2\right) = \sum_{\substack{J_0 \in \mathcal{I}(S) \\ n^{\text{th}}\text{-power free}}} \frac{\epsilon(J_0)\psi_2(J_0) L_S(\frac{1}{2}, \pi, \overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J_0)^w} \sum_{J_n \in \mathcal{I}(S)} \frac{Q(\frac{1}{2}, J; \pi, \overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J_n)^{nw}}.$$

In the expression for $Q(\frac{1}{2}, J; \pi, \overline{\chi}_{J_0}\psi_1)$ obtained from (2.11), we abbreviate the notation by defining

$$\mathcal{C}_{J_0,\psi_1}(P_v) = \overline{\chi}_{J_0}(P_v)\psi_1(P_v) + \chi_{J_0}(P_v)\overline{\psi}_1(P_v), \tag{6.1}$$

and writing

$$\mathcal{Q}\left(\frac{1}{2}, P_v^{n\gamma}, \overline{\chi}_{J_0}\psi_1; \pi\right) = a(P_v^{n\gamma}) - a(P_v^{n\gamma-1}) q_v^{-\frac{1}{2}} \mathcal{C}_{J_0,\psi_1}(P_v) + a(P_v^{n\gamma-2}) q_v^{-1}.$$

For fixed J_0, we show that the sum over J_n is absolutely convergent in a certain right half-plane in w. We may write

$$\sum_{J_n \in \mathcal{I}(S)} \frac{Q(\frac{1}{2}, J; \pi, \overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J_n)^{nw}}$$
$$= \prod_{k=1}^{n-1} \prod_{\substack{v \\ \mathrm{ord}_v(J) = n\gamma + k}} \left(\sum_{\gamma \geq 0} \frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{k-1})}{(\mathbb{N}P_v)^{n\gamma w}} \right) \prod_{\substack{v \\ \mathrm{ord}_v(J) = n\gamma}} \left(\sum_{\gamma \geq 0} \frac{\mathcal{Q}(\frac{1}{2}, P_v^{n\gamma}, J; \pi)}{(\mathbb{N}P_v)^{n\gamma w}} \right)$$

and then analyze each of the geometric sums individually, using the Satake parameters. First, we have

$$\begin{aligned}\sum_{\gamma\geq 0}\frac{a(P_v^{n\gamma+k-1})\psi_1(P_v^{k-1})}{(\mathbb{N}P_v)^{n\gamma w}} &= \frac{\psi_1(P_v^{k-1})}{\alpha_v-\beta_v}\sum_{\gamma\geq 0}\frac{\alpha_v^{n\gamma+k}-\beta_v^{n\gamma+k}}{q_v^{n\gamma w}}\\ &= \frac{\psi_1(P_v^{k-1})}{\alpha_v-\beta_v}\left[\alpha_v^k(1-\alpha_v^n q_v^{-nw})^{-1}-\beta_v^k(1-\beta_v^n q_v^{-nw})^{-1}\right]\\ &= \psi_1(P_v^{k-1})(1-\alpha_v^n q_v^{-nw})^{-1}(1-\beta_v^n q_v^{-nw})^{-1}\left[a(P_v^{k-1})+a(P_v^{n-k-1})q_v^{-nw}\right].\end{aligned}$$

Notice that we have two of the factors in the Euler factor corresponding to the non-archimedean places v in the symmetric n^{th} power L-function. Namely, for each such place v, we have the $j=0$ and $j=n$ factors in the expression

$$L^{(v)}(nw,\pi,\mathrm{sym}^n)=\prod_{0\leq j\leq n}(1-\alpha_v^{n-j}\beta_v^j q_v^{-nw})^{-1}. \tag{6.2}$$

Next, we have

$$\begin{aligned}\sum_{\gamma\geq 0}\frac{\mathcal{Q}(\frac{1}{2},P_v^{n\gamma},\overline{\chi}_{J_0}\psi_1;\pi)}{(\mathbb{N}P_v)^{n\gamma w}} &= 1+\sum_{\gamma\geq 1}\frac{a(P_v^{n\gamma})-a(P_v^{n\gamma-1})q_v^{-\frac{1}{2}}\mathcal{C}_{J_0,\psi_1}(P_v)+a(P_v^{n\gamma-2})q_v^{-1}}{q_v^{n\gamma w}}\\ &= 1+\frac{1}{\alpha_v-\beta_v}\Bigg[\left(\alpha_v-q_v^{-\frac{1}{2}}\mathcal{C}_{J_0,\psi_1}(P_v)+(\alpha_v q_v)^{-1}\right)\frac{\alpha_v^n}{q_v^{nw}}(1-\alpha_v^n q_v^{-nw})^{-1}\\ &\qquad +\left(\beta_v-q_v^{-\frac{1}{2}}\mathcal{C}_{J_0,\psi_1}(P_v)+(\beta_v q_v)^{-1}\right)\frac{\beta_v^n}{q_v^{nw}}(1-\beta_v^n q_v^{-nw})^{-1}\Bigg].\end{aligned}$$

After factoring out $(1-\alpha_v^n q_v^{-nw})^{-1}(1-\beta_v^n q_v^{-nw})^{-1}$ from the entire expression and simplifying, we obtain

$$\begin{aligned}&\sum_{\gamma\geq 0}\frac{\mathcal{Q}(\frac{1}{2},P_v^{n\gamma},\overline{\chi}_{J_0}\psi_1;\pi)}{(\mathbb{N}P_v)^{n\gamma w}}\\ &\quad=(1-\alpha_v^n q_v^{-nw})^{-1}(1-\beta_v^n q_v^{-nw})^{-1}\left[1+\frac{a(P_v^{n-2})}{q_v^{nw}}-\frac{a(P_v^{n-1})\mathcal{C}_{J_0,\psi_1}(P_v)}{q_v^{nw+1/2}}+\frac{a(P_v^{n-2})}{q_v^{nw+1}}+\frac{1}{q_v^{2nw+1}}\right].\end{aligned}$$

Therefore the sum over $J_n\in\mathcal{I}(S)$ becomes

$$\begin{aligned}&\sum_{J_n\in\mathcal{I}(S)}\frac{Q(\frac{1}{2},J;\pi,\overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J_n)^{nw}}= \\ &\prod_v(1-\alpha_v^n q_v^{-nw})^{-1}(1-\beta_v^n q_v^{-nw})^{-1}\prod_{k=1}^{n-1}\prod_{\substack{v\\ \mathrm{ord}_v(J)\equiv k\,(n)}}\psi_1(P_v^{k-1})\left[a(P_v^{k-1})+a(P_v^{n-k-1})q_v^{-nw}\right]\\ &\qquad\times\prod_{\substack{v\\ \mathrm{ord}_v(J)\equiv 0\,(n)}}\left[1+\frac{a(P_v^{n-2})}{q_v^{nw}}-\frac{a(P_v^{n-1})\mathcal{C}_{J_0,\psi_1}(P_v)}{q_v^{nw+1/2}}+\frac{a(P_v^{n-2})}{q_v^{nw+1}}+\frac{1}{q_v^{2nw+1}}\right].\end{aligned} \tag{6.3}$$

In order to express this sum in terms of $L(nw, \pi, \mathrm{sym}^n)$, we multiply through by the remaining factors in (6.2) and their reciprocals. If we put

$$R_v(n, w; \pi) = \prod_{1\le j\le n-1} (1 - \alpha_v^{n-j}\beta_v^j q_v^{-nw})$$
$$= \begin{cases} (1-\alpha_v^{n-2}q_v^{-nw})\cdots(1-\alpha_v q_v^{-nw})(1-\beta_v q_v^{-nw})\cdots(1-\beta_v^{n-2}q_v^{-nw}) & \text{if } n \text{ is odd,} \\ (1-\alpha_v^{n-2}q_v^{-nw})\cdots(1-\alpha_v q_v^{-nw})(1-q_v^{-nw})(1-\beta_v q_v^{-nw})\cdots(1-\beta_v^{n-2}q_v^{-nw}) & \text{if } n \text{ is even,}\end{cases}$$

then we have

$$\sum_{J_n\in\mathcal{I}(S)} \frac{Q(\frac{1}{2}, J; \pi, \overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J_n)^{nw}} = L_S(nw, \pi, \mathrm{sym}^n)\,\mathcal{R}_{J_0}(w;\pi),$$

where

$$\mathcal{R}_{J_0}(w;\pi) = \prod_{k=1}^{n-1}\prod_{\substack{v\\ \mathrm{ord}_v(J_0)=k}} R_v(n,w;\pi)\psi_1(P_v^{k-1})\left[a(P_v^{k-1}) + a(P_v^{n-k-1})q_v^{-nw}\right]$$
$$\times \prod_{\substack{v\\ \mathrm{ord}_v(J_0)=0}} R_v(n,w;\pi)\left[1 + \frac{a(P_v^{n-2})}{q_v^{nw}} - \frac{a(P_v^{n-1})\mathcal{C}_{J_0,\psi_1}(P_v)}{q_v^{nw+1/2}} + \frac{a(P_v^{n-2})}{q_v^{nw+1}} + \frac{1}{q_v^{2nw+1}}\right].$$

The factor $L(nw, \pi, \mathrm{sym}^n)$ converges absolutely for $\mathrm{Re}(w) > \frac{1}{n} + \frac{1}{9}$. In the factor $\mathcal{R}_{J_0}(w;\pi)$, the product over places v with $P_v \mid J_0$ is a finite product, and therefore it does not affect convergence. In the infinite product over places v with $P_v \nmid J_0$, for a given place v, it is clear that the terms

$$\frac{a(P_v^{n-2})}{q_v^{nw}} \qquad \text{and} \qquad \frac{a(P_v^{n-1})\mathcal{C}_{J_0,\psi_1}(P_v)}{q_v^{nw+1/2}}$$

determine the region of convergence. Using the fact that

$$a(P_v) \ll_\epsilon q_v^{1/9+\epsilon},$$

([12]) we see that the first of these two terms is in fact more restrictive. We find that this infinite product, and hence $\mathcal{R}_{J_0}(w;\pi)$, converges absolutely for $\mathrm{Re}(w) > \frac{7}{9n} + \frac{1}{9}$. Now suppose there are only finitely many twists for which $L_S(\frac{1}{2}, \pi, \overline{\chi}_{J_0}\psi_1)$ is nonzero. Then

$$L_S(nw, \pi, \mathrm{sym}^n) \sum_{\substack{J_0\in\mathcal{I}(S)\\ n^{\text{th}}\text{-power free}}} \frac{\epsilon(J_0)\psi_2(J_0)L_S(\frac{1}{2}, \pi, \overline{\chi}_{J_0}\psi_1)}{(\mathbb{N}J_0)^w}\,\mathcal{R}_{J_0}(w;\pi),$$

will converge absolutely for $\mathrm{Re}(w) > \frac{1}{n}+\frac{1}{9}$. (Note that since we are restricting our attention to the case $n \ge 3$, this would mean that there exists some $\delta > 0$ such that the sum converges absolutely for $\mathrm{Re}(w) > \frac{1}{2} - \delta$.)

7. Nonvanishing twists (Proof of Theorem 1.1)

We now use the results of the previous sections to prove Theorem 1.1. We require the following lemma.

Lemma 7.1. *Suppose the Dirichlet series*

$$L(w) = \sum \frac{b(d)}{d^w}$$

is absolutely convergent for $Re(w) > 1/2 - \delta$, *for some positive* δ. *Suppose further that there exist Dirichlet series* $M_1(w), M_2(w), \ldots, M_r(w)$ *and functions* $\gamma_1(w), \gamma_2(w), \ldots, \gamma_r(w)$ *which satisfy the following conditions:*

(1) *Each* $M_j(w)$ *is absolutely convergent for* $Re(w) > 1/2 - \delta$.

(2) *Each* $\gamma_j(w)$ *is holomorphic for* $Re(w) > 0$, *and for all* $k > 0, \sigma > 1/2$ *we have the estimate*

$$\gamma_j(\sigma + it) <<_{k,\sigma} |t|^{-k}, \text{ as } |t| \to \infty.$$

(3) *There is the functional equation*

$$L(w) = \sum_j \gamma_j(w) M_j(1-w).$$

Then $L(w)$ *is identically zero.*

To apply the Lemma, we set $s = \frac{1}{2}$ and view the functions $Z_i(s, w; \pi, \psi_1, \psi_2)$ for $i = 1, 2$ as Dirichlet series in w. In particular, after repeated applications of the functional equations, the Dirichlet series $L(w) = Z_1(\frac{1}{2}, w; \pi, 1, 1)$ satisfies the functional equation

$$L(w) = \sum_{\xi_1, \xi_2 \in \widehat{R}_c} \gamma_{\xi_1,\xi_2}(w) Z_2(1/2, 1-w; \pi, \xi_1, \xi_2)$$

for some collection of functions γ_{ξ_1,ξ_2} satisfying condition 2 of Lemma 7.1. (To see this, apply the w-functional equation, followed by the s-functional equation, followed by the w-functional equation to $Z_1(s, w; \pi, 1, 1)$ at $s = \frac{1}{2}$.) If there are only finitely many idèle class characters χ_{J_0} of order n such that $L(1/2, \pi \otimes \chi_{J_0})$ is nonzero, then as established in the previous section, there exists a positive δ such that the Dirichlet series on both sides of the above equation are absolutely convergent for $\mathrm{Re}(w) > 1/2 - \delta$. It follows that $L(w)$ is identically zero and the proof of Theorem 1 is complete.

Proof of Lemma 7.1: Suppose $L(w)$ is not identically zero. Choose δ' with $0 < \delta' < \delta$ such that $L(1/2+\delta') = A \neq 0$. Then

$$L(1/2+\delta'+it) \ll \sum_j |\gamma_j(1/2+\delta'+it)| \cdot |M_j(1/2-\delta')| \longrightarrow 0 \qquad \text{as } |t| \to \infty.$$

Choose X so large that $\left| L(1/2+\delta') - \sum_{d<X} \frac{b(d)}{d^{1/2+\delta'}} \right| < A/3$. Choose t_0 so that $\left| d^{-it_0} - 1 \right| < 1/3$ for all $d < X$. Then

$$|L(1/2+\delta'+it_0) - L(1/2+\delta')| \leq \left| \sum_{d<X} \frac{b(d)}{d^{1/2+\delta'}} \left(1 - \frac{1}{d^{it_0}}\right) \right| + \left| \sum_{d \geq X} \frac{b(d)}{d^{1/2+\delta'}} \right| < 2A/3.$$

Hence $|L(1/2+\delta'+it_0)| > A/3$. However, we can find arbitrarily large such t_0. This contradicts the fact that $L(1/2+\delta'+it) \to 0$ as $t \to \infty$. $\square$

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